Trisecting an angle with compass, straight edge, and wisdom from Dr. Covey and Captain Kirk.

TL;DR: In science "practice" is superior to "theory". This paper describes a construction with four or six angle bisections, one line trisection, and a few lines to trisect an angle. In 1876 M. Wantzel proffered a negative proof stating "[trisecting an angle] … cannot be solved in general with ruler and compass." More appropriate might have been "... general using the equation above." which allows for alternatives. See *Allegory,* endnote *7* on page 89. If you understand the barriers to trisection you might want to first see Page 14 of *Appendix B, How Accurate (Magnified)* on page 38. [98 words]

ACADEMIC TITLE:

Implications of a small Sagitta-Short and a large Arc Radius on a Euclidean Angle Trisection

ABSTRACT: Some historical statements widely accepted as fact and later proven wrong. These include "The Sun revolves around the Earth", "Only birds can fly", and "We'll never get to the moon". In 1836 Wantzel proved trisecting an angle could done via specific equations (true), and stated since most of those equations could not be solved with straightedge and compass, neither could the trisection problem. The latter is a logical fallacy excluding the possibility of a solution without such equations. His "negative proof" essentially ended trisection work by "serious" mathematicians. There is an effective construction with roots in a grade school math problem, Dr. Covey's "Start with the end in mind", and Captain Kirk's "I changed the conditions of the test." [119 words]

KEYWORDS: geometry, angle, trisection, tri-section, v105, logic, euclid, wantzel, covey, kirk, ants, 52, science, theory, practice

AUTHOR: Jonathan E. Jaffe, CPA, MS an auditor, consultant, teacher, inventor, author, and not a professional academic OrcID: 0009-0007-8806-0209

CONTACT DETAILS: jejaffe@nc3.mobi https://www.linkedin.com/in/jonathanejaffe https://orcid.org/0009-0007-8806-0209

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Table of Contents

License

Introduction To Construction

Widely Held

Who Founded Geometry?
It is widely held that Euclid was the founder of geometry[1] circa 300 BC. Yet Babylonians were using geometry and trigonometry for land surveys as early as 1900 BC[2] [3].

Unsolved Problems of Geometry
It is widely held that one of unsolved *The Three Famous Problems of Geometry* is trisecting an angle by traditional Euclidean geometry[4].

Proven Unsolvable
It is widely held that in 1836 Pierre Laurent Wantzel proved[5] trisecting an angle could be solved via a cubic equation (true) and since most cubic equations could not be solved with ruler (unmarked straightedge) and compass, neither could trisection be solved (a logical fallacy[6]). It is generally accepted you can't prove a negative. See an allegory[7] for a simpler example of the fallacy.

That some problem can't be solved in one manner does not preclude another manner to solution. It was widely held that the great pyramids of Egypt were built from quarried sandstone. Using formed sandstone slurry was considered almost heretical. Circa 1987 the use of slurry was proven[8].

Four Tools, Not Two
Per Wantzel[9] it is not possible to trisect an angle of any given value with a straightedge and compass. Two other tools are required: a writing implement and something on which to write. A stick and an area of earth were both were available circa 300 BC for Euclid and circa 1900 BC for Babylonians. The omission lead to a recollection of "A difference that makes no difference is no difference."

Yet

In our known history some statements have been widely held and later proven false. A few examples:

Land Speed
"Man will never travel faster than a good horse." Steam powered rail travel was first used in Great Britain in 1825[10] and was improved for a longer sustained duration[11] at a speed faster than horses.

Manned Flight
It was widely held that flying was reserved for insects and avians. Circa 1500 Leonardo da Vinci sketched flying machines[12]. On December 17, 1903 a free, controlled flight of a power-driven, heavier-than-air, aircraft was aloft for 59 seconds and traveled over 850 feet[13].

No Escape
No one can escape from Luft Stalag III. 600$^+$ prisoners built three tunnels. Over 70 prisoners escaped[14].

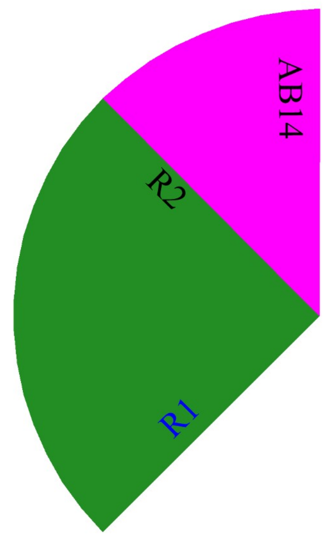
Space Flight
"We'll never get to the moon!" In 1948 "Albert I" went to space[15]. In 1961 astrochimp "Ham[16]", cosmonaut Gagarin, and astronaut Shepard all went. Projects Mercury and Gemini[17] were learning how to do more. 1968 Apollo 8[18] orbited the moon. Just 21 years after Albert I humans walked on the moon during the Apollo 11[19] mission in 1969.

Suffix Required
There are many other examples. When someone says "It isn't possible to [do something]" one word should be routinely added … YET.

Science Duality

Science is often divided into *Theory and Practice,* sometimes *Theoretical and Observational*, or other similar terms. If repeated and controlled practice, or observation, contradicts theory then the theory is generally changed.

What if something can be practiced / observed / constructed and theory says it can't be done? That isn't a hypothetical happenstance. See *Yet* on page 4 for several examples where theory was contradicted by practice. Add this: "No one will ever need more than 64k on their computer."



Half the problem

Roots of Solution
This solution has roots in a grade school math problem[20], re-envisioned with Dr. Covey's "Start with the end in mind"[21] and Captain Kirk's "I changed the conditions of the test[22]." The construction starts with half the problem.

Writing Tools

Ancient
The chemistry of cave paintings[23] is well known. Early brushes[24]? Not so much.

Reed & Quill
Early pens were thin reed brushes or reed pens[25] writing on papyrus about 4th Century BC[26], prior to Euclid. These were generally supplanted by quill pens[27] with replaceable nibs for a variety of line widths. Circa 2010 nibs had a reported line width as fine (narrow) as 0.1 mm[28].

Education

Includes Unreality
Understanding new concepts is easier when conveyed in small bits and a simple manner. In early physics teachers refer to ropes without the complication of mass, pulleys without friction, electricity in wire moving at the speed of light, and similar fictions[29].

Line Types
In mechanical drawing (high school) and engineering graphics (university) we were taught that line width was important.[30] In a creation requiring a high level of detail and complexity these manual disciplines have gradually given way to, or become a basis for, computer assisted design (CAD).

Hairlines
CAD has a line type called "hairline", the thinnest line you can see and it stays that way. Even at screen magnification of over 10,000% (100x) it still appears as thin a line as it did at 100% (1x). When you zoom out you can still see a hairline … but only so far before it isn't visible at all. While hairlines are convenient for making connections and constructions they can create a misleading mindset when considering reality. The drawings in this presentation often started with hairlines and include magnified versions for better viewing. The presentation also includes drawings where the line width is scaled to real-world width for a significant difference in appearance and implications.

Geometric Constructions

Bisection Won't Get Trisection
Bisection of the largest angle 360-degrees gets two 180-degree angles. Continuing we get 90, 45, 22.5, 11.25, 5.625, and so on, all discrete[31] numbers. The fraction 1/3 can be expressed as a never-ending decimal 0.33333etc. 2/3 is decimal 0.66666etc. Dividing a discrete number by two won't create a recurring or repeating decimal[32]. Bisection can get closer to a trisection solution.

Infinitesimal Sagitta-Short
A circle may be considered as multiple joined line sections each so short they are indistinguishable from a continuous curve. So what happens when a chord has a sagitta-short (see page 26 in *Appendix B* for sagitta definitions) so small as to be infinitesimal[33]? How about so small that a 0.1 mm line weight used for that arc and for that chord are indistinguishable from each other and together obscure the sagitta? The implications are described in *Appendix B – Infinitesimal Geometry* starting on page 25.

Presentation

Format Notes

Page Orientation
The drawing pages are presented landscape. The wider presentation allows some small constructions to be better visualized for easier understanding. Modern computer monitors are wider than tall. Pad, tablet, cell phone screens, etc have landscape display modes.

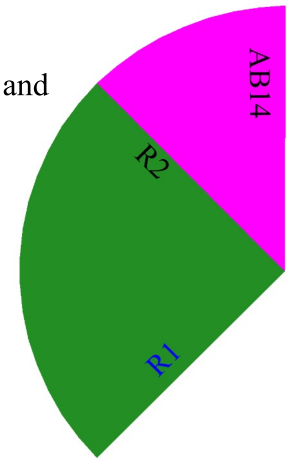
Creation Tools Used
The majority of the narrative here and above was created in a word processor. The drawings that follow were created in computer assisted design with text and image elements.

Scaling & Appearance
CAD drawings were made as 1:1 in model space. That is: one inch was scaled at one inch. Zooming in (magnification) allow smaller distances to appear larger for easier understanding. Zooming out makes large distances appear smaller. Some zoom was necessary to fit CAD elements within word processing borders in both portrait and landscape orientations.

Appendix Order
"Appendix A" many not be referenced first.

Presentation

Summary



The problem is most commonly presented as a single angle formed with two rays, R1 and R4, sharing a common vertex V. Working with half the angle presented and an arc provided a new view and a potential solution: Half the angle Ray 1 – Vertex – Ray 4 (R1-V-R4) is marked by AB14, the angle bisector of Ray 1 and Ray 4.

The distance along an arc from R1 to AB14 is divided at R2 where the distance along that arc from R1 to R2 is twice the distance as from R2 to AB14.

Consider the travels of two ants where Ant-A travels along an arc, centered at V, starting at R1 and moves toward AB14. Ant-B travels along the same arc starting at AB14 and moves toward R1. Ant-B moves at a speed of 1x. Ant-A moves at 2x. They will meet along the arc at R2[34]. R3 is created by mirroring R2 about AB14 to show both trisection rays.

Appendices follow the presentation below.

Implications of a small Sagitta-Short and a large Arc Radius on a Euclidean Angle Trisection

alternative title:

Missed it by that much

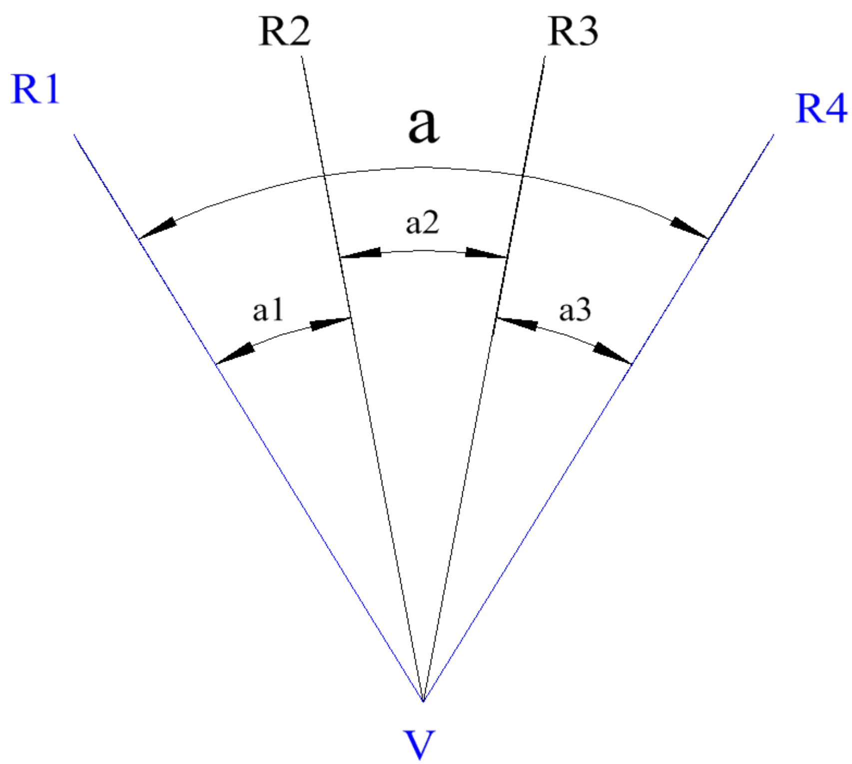
meets

Infinitesimal Geometry

alternative to alternative title:

Travels of Ant A and Ant B

35



Blue generally indicates an element given in the problem. Here the line and label for R1 are given. The same for R4 and the vertex V.

Part 1 page 1 - The Solution

Where

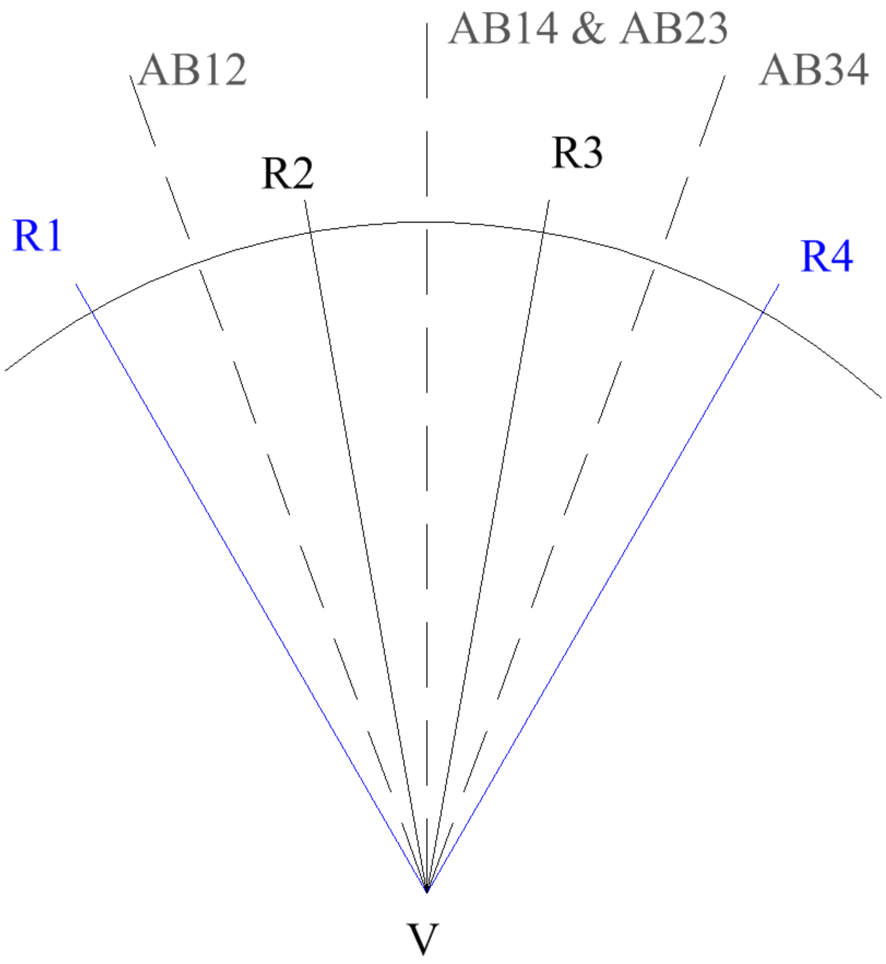
angle a is trisected into three equal angles the sum of which matches the original angle.

prefix a refers to an angle
prefix R refers to a Ray
eminating from V the vertex

a = R1-V-R4
a1 = R1-V-R2
a2 = R2-V-R3
a3 = R3-V-R4

and
a1 = a2 = a3

and
a1 + a2 + a3 = a



Notice the angle bisectors (ABnn) and the rays are symmetrical about AB14 (pronounce as ei-bee-one-four, not Abb-fourteen) and AB23 (æ-bee-two-three, not Abb-twentythree)

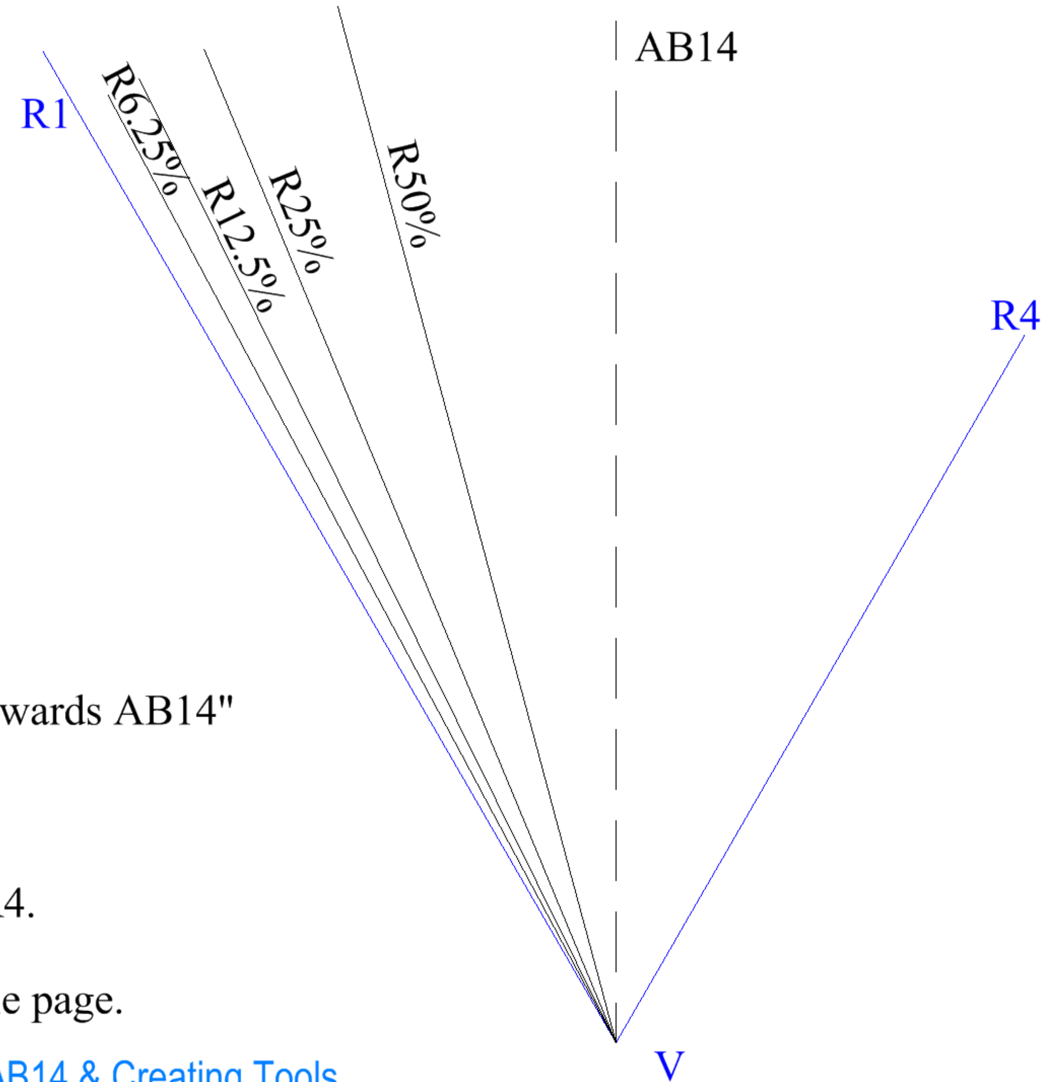
AB14 is the bisector of R1-V-R4 and can be derived from the given problem.

The locations of R2 and R3 and their bisectors are not known from the given problem.

They are shown here at the solution
AB12 is the bisector of R1-V-R2
AB23 is the bisector of R2-V-R3
AB34 is the bisector of R3-V-R4

AB23 and AB14 share the same line.

Part 1 page 2 - Derived at the Solution - Angle Bisectors



Create AB14 as the angle bisector of angle R1-V-R4

Appendicies are not necessarily referenced in order.

Create the tools per Appendix C.

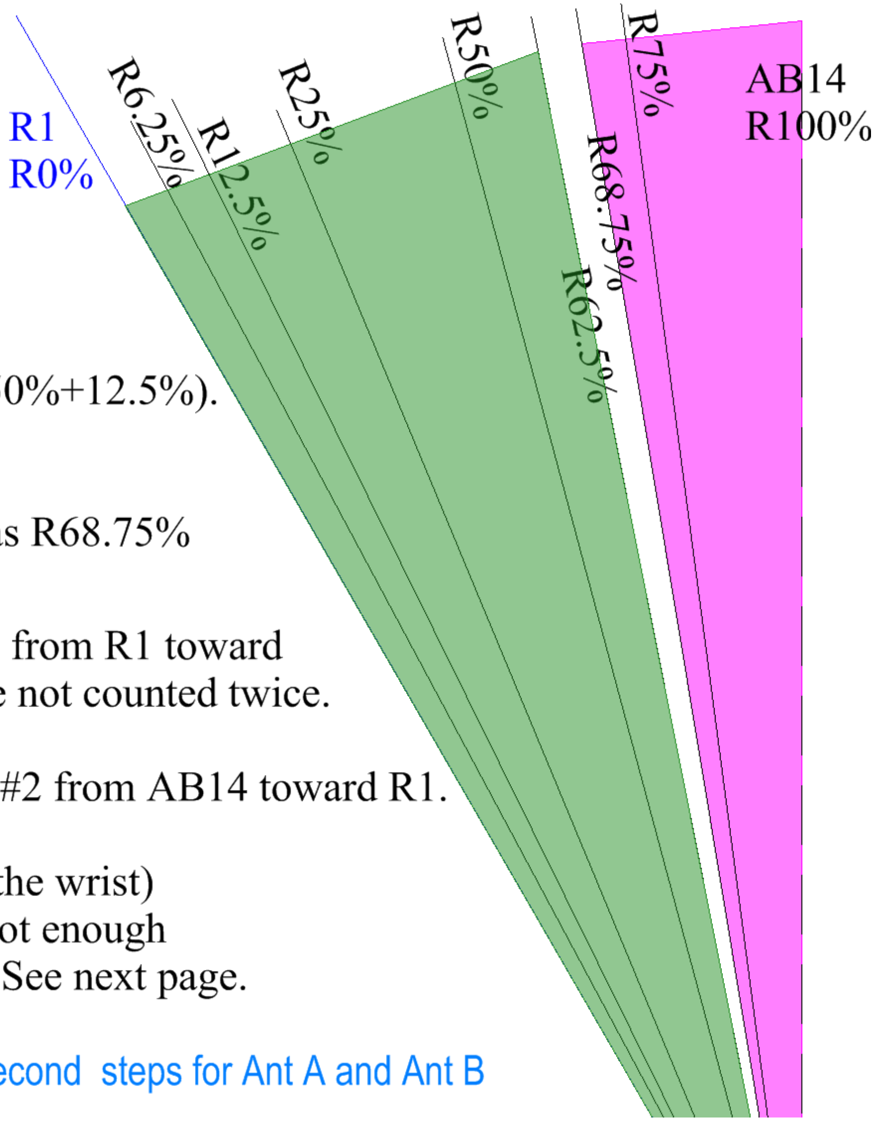
R25% is a short notation for
"Ray at 25% of the arc from R1 towards AB14"
Similar for all Rnn% labels

On the next page we'll only be viewing the half the angle R1-V-R4.

The vertex moves below the visible page.

Part 2 - Building a Solution - page 1 - AB14 & Creating Tools

In the tools created each smaller angle is half the next angle clockwise. R25% is the bisector of R1-V-R50%. R12.5% is the bisector of R1-V-R25% etc. Create only to R6.25% for now.



Ant-A step #1 already present as R50%.

Ant-B step #1 Copy angle R1-V-R25% COUNTERCLOCKWISE from R100% (AB14). Label that R75% (100%-25%).

Ant-A step #2 - Add angle R0%-V-R12.5% CLOCKWISE to R50%. Label as R62.5% (50%+12.5%).

Ant-B step #2 - Add angle R0%-V-R6.25% COUNTER CLOCKWISE to R75%. Label as R68.75%

Green zone shows Ant A step #1 and step #2 from R1 toward AB14. Tools in zone from R0%-V-R50% are not counted twice.

Magenta zone shows Ant B step #1 and step #2 from AB14 toward R1.

Next step is to fit the last pad (furthest from the wrist) of the little finger to R62.5%-V-R68.75%. Not enough room so I will move further from the vertex. See next page.

Part 2 - Building a Solution - page 2 - First and Second steps for Ant A and Ant B

Part 2 – Finger Pad & Arc-a
Notice that R1 is also R0% and AB14 is also R100%.

For clarity labels and tools no longer required are not shown.

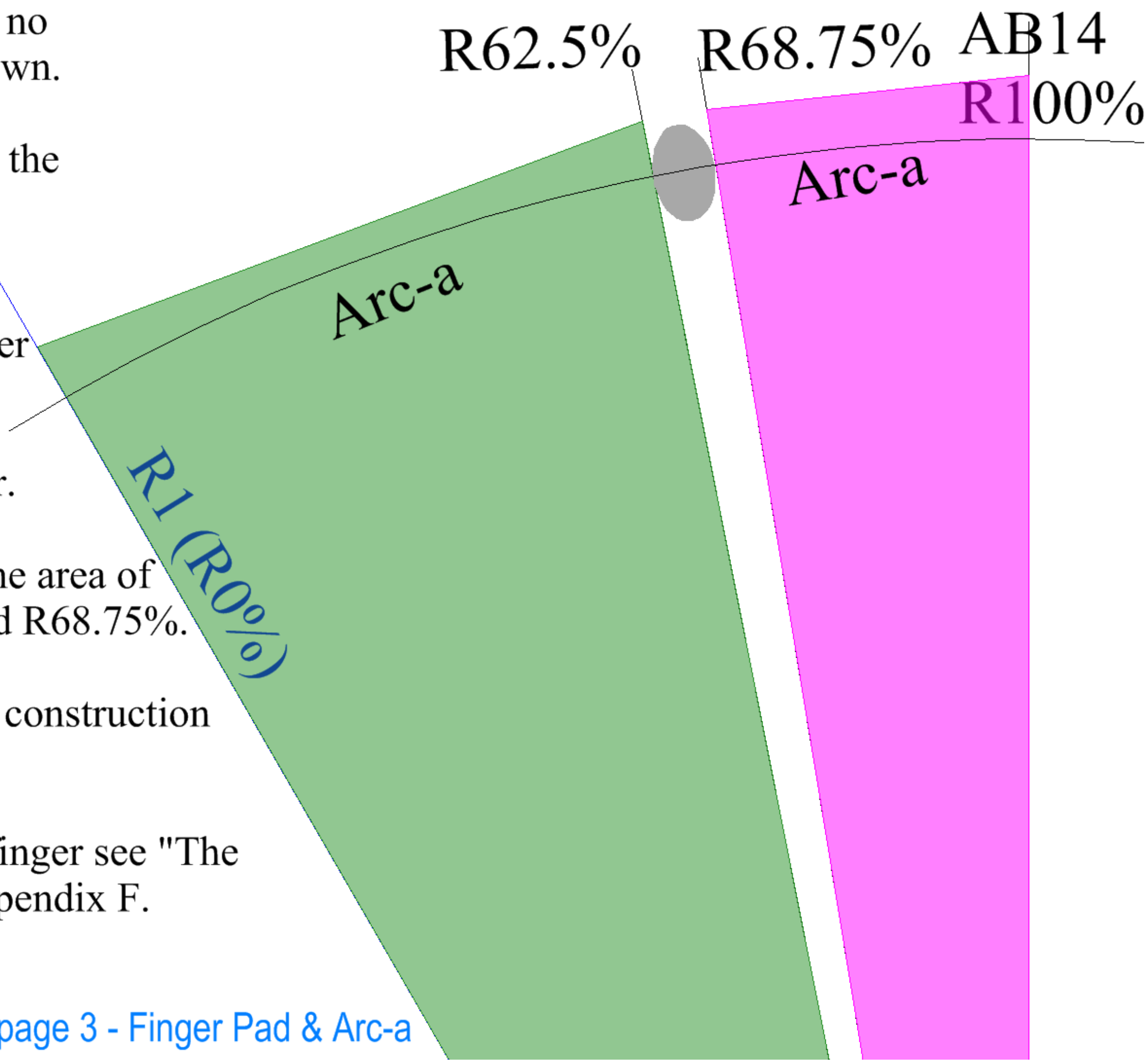
The last pad (furthest from the wrist) of the little finger is shown in gray.

Now draw Arc-a with center at V (not visible in this drawing) and a radius to the wide point of the finger.

For clarity we'll magnify the area of Arc-a between R62.5% and R68.75%.

Later we'll show the entire construction on a single page.

If you don't want to use a finger see "The Third Pair of Tools" in Appendix F.



Part 2 - Building a Solution - page 3 - Finger Pad & Arc-a

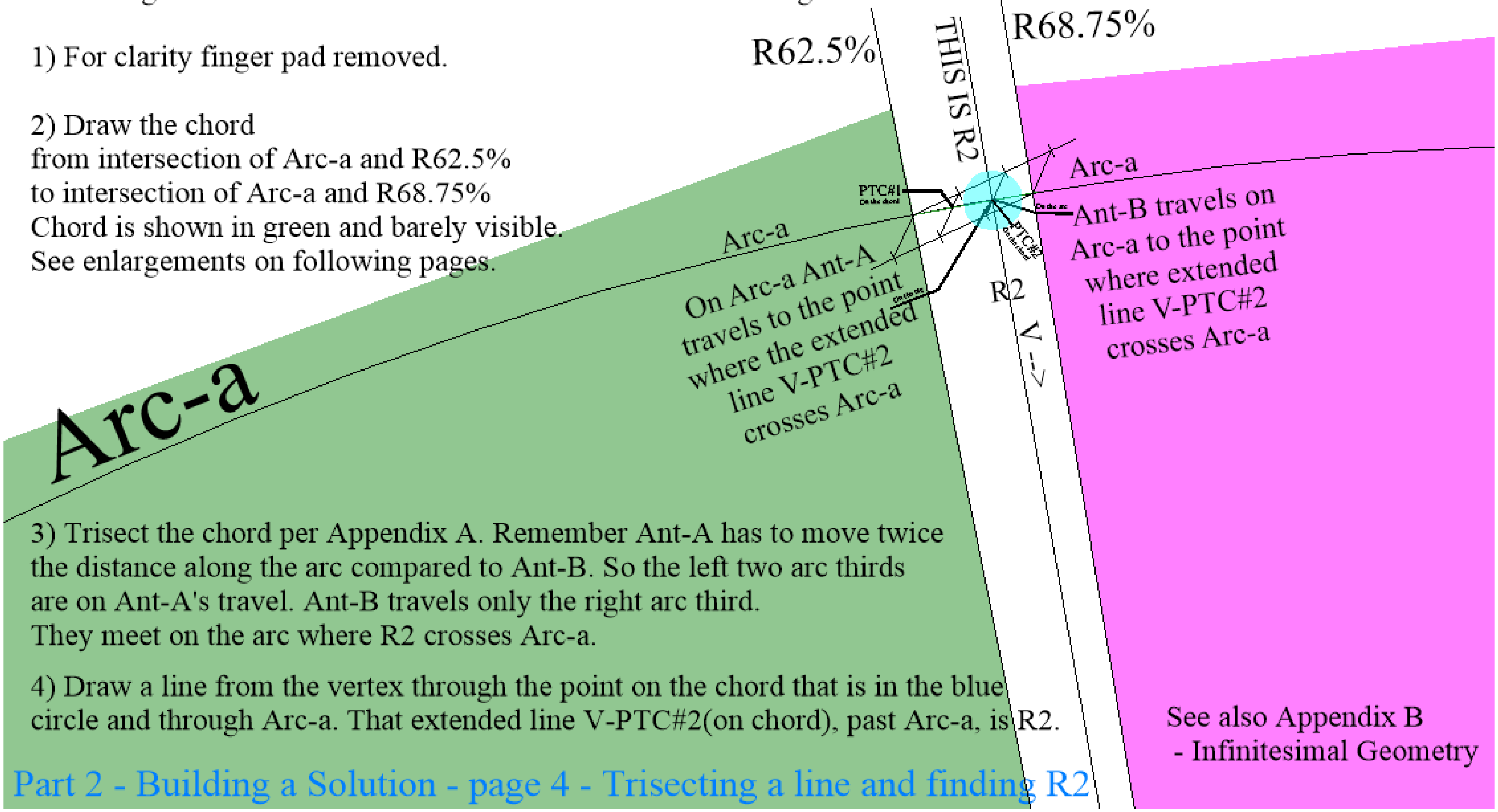
Part 2 – TrisectChord>PTC#2>R2

Nomenclature: PTC is Point-Trisection-Chord where PTC#1 is further from AB14 than PTC#2.

Points PTC#1 and PTC#2 mark the equal thirds of the chord.
Lines V-PTC#1 and V-PTC#2 are rays that trisect the chord.
Extending line V-PTC#2 farther from V trisects the arc running from R1 to AB14.

1) For clarity finger pad removed.

2) Draw the chord
from intersection of Arc-a and R62.5%
to intersection of Arc-a and R68.75%
Chord is shown in green and barely visible.
See enlargements on following pages.

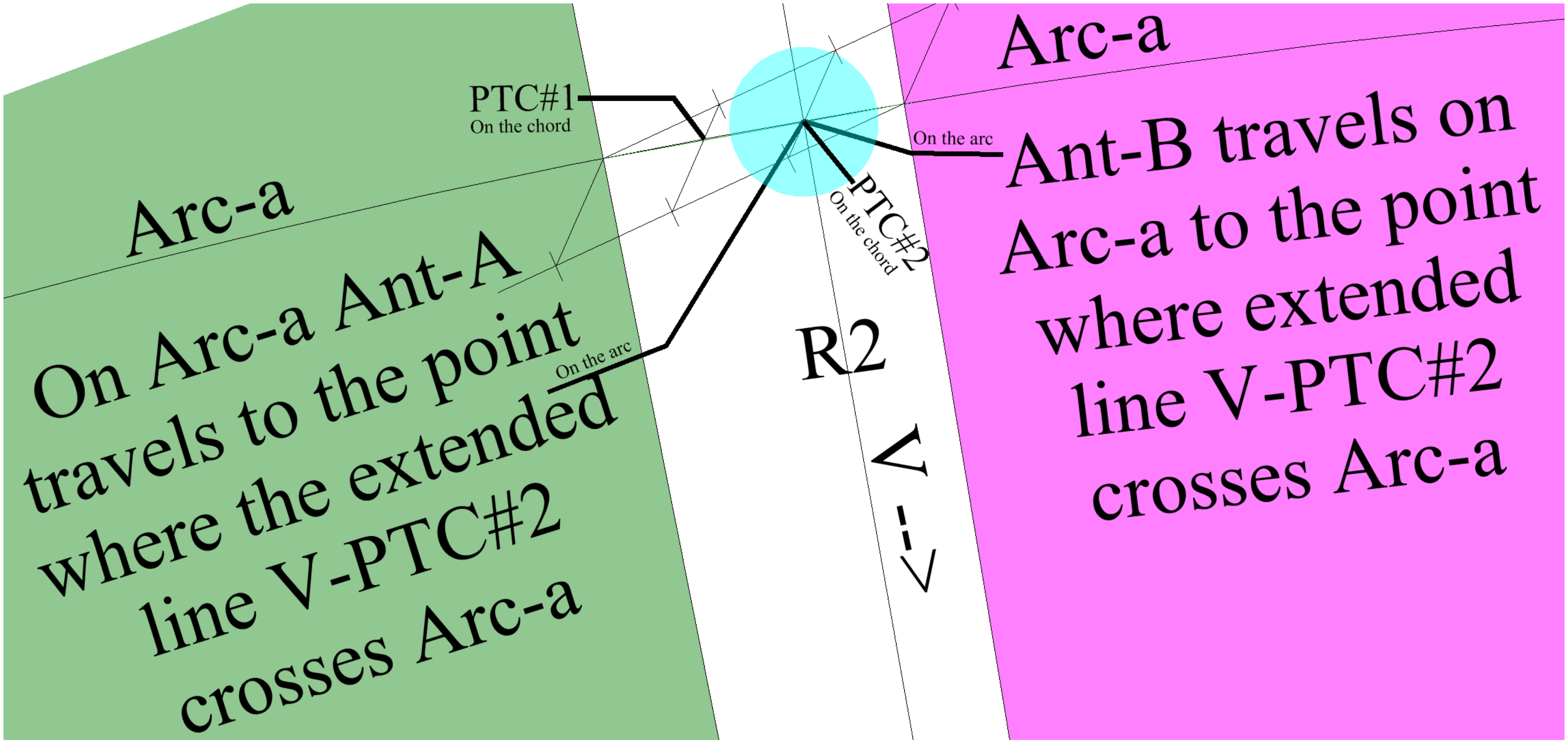


3) Trisect the chord per Appendix A. Remember Ant-A has to move twice the distance along the arc compared to Ant-B. So the left two arc thirds are on Ant-A's travel. Ant-B travels only the right arc third.
They meet on the arc where R2 crosses Arc-a.

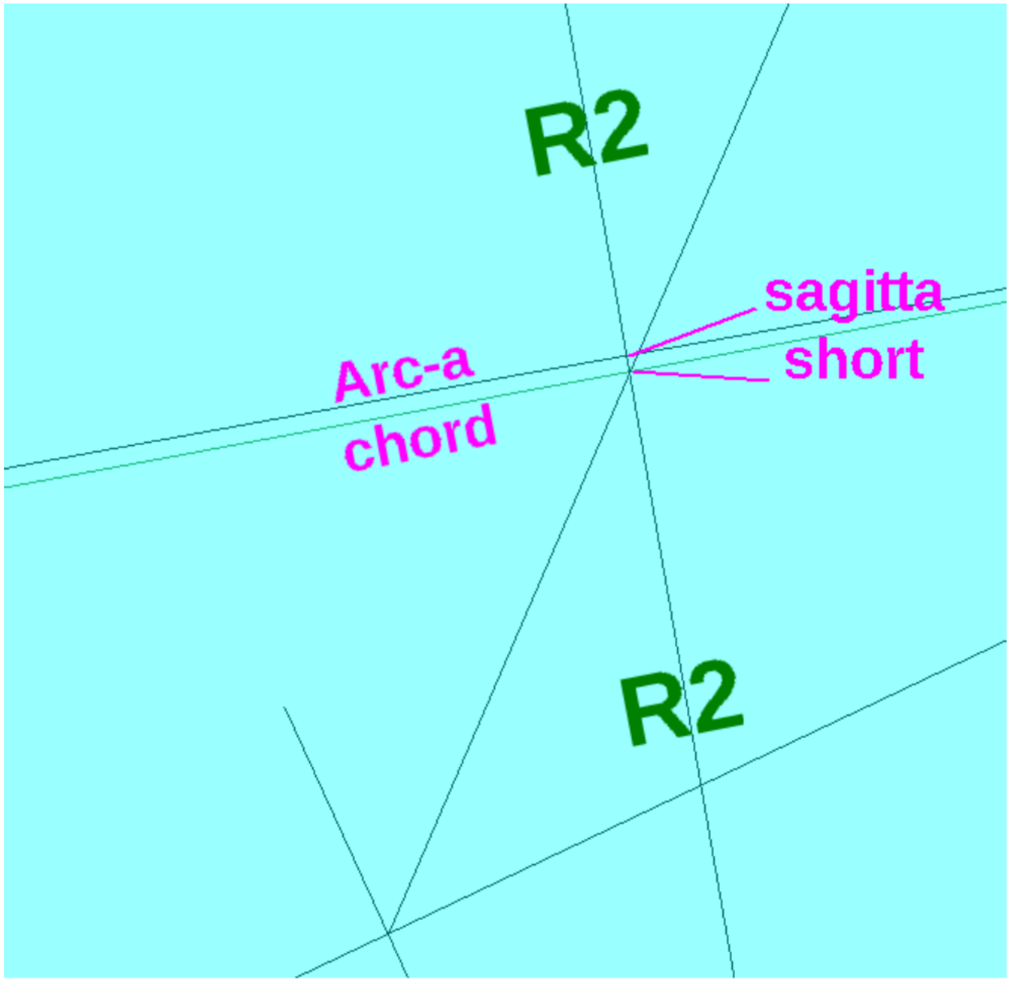
4) Draw a line from the vertex through the point on the chord that is in the blue circle and through Arc-a. That extended line V-PTC#2(on chord), past Arc-a, is R2.

See Appendix G on page 68 for an optional technique that provides more paper space for chord trisection.

Part 2 – Chord (magnified)



Part 2 – Finding R2 (magnified)



The finger pad width was measured at 0.352554 inches and that was also the chord length.

The radius of Arc-a was measured at 10.441581", larger than the paper, because the finger pad would not initially fit between R62.5% and R68.75%.

The sagitta short was measured and calculated to 0.037797 mm.

That is 7.5594% of a 0.5mm pencil lead.

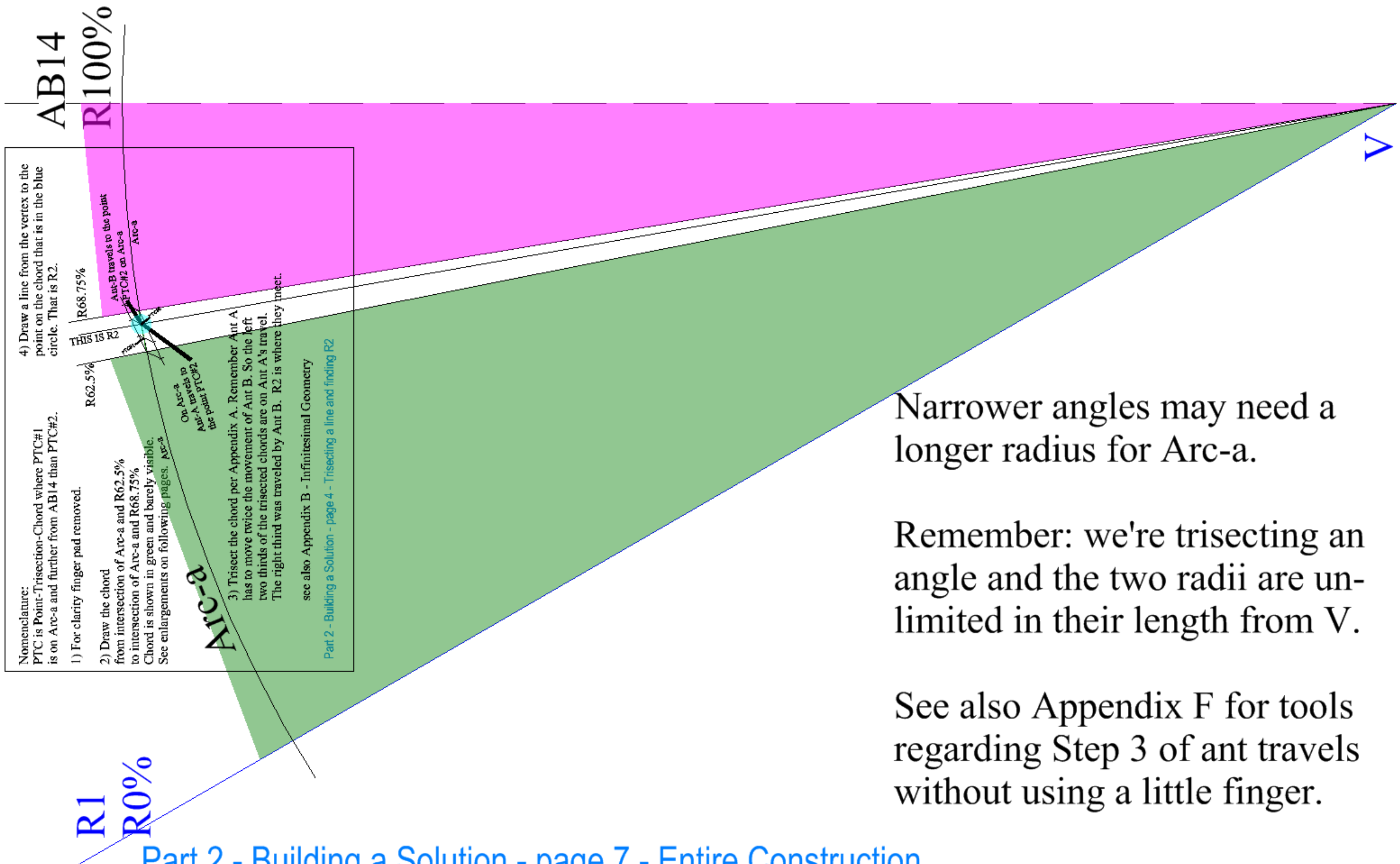
That is 37.797% of a 0.1mm pencil lead.

Both are infinitesimal.

Part 2 - Building a Solution - page 6 - Finding R2 (enlarged)

A 0.1 mm line width, centered on the chord and centered on the arc, would obscure the sagitta-short. For a magnified image of 0.1mm and 0.5mm line widths obscuring a sagitta-short see Appendix B hairlines on page 33.

Part 2 – Entire Construction



Narrower angles may need a longer radius for Arc-a.

Remember: we're trisecting an angle and the two radii are unlimited in their length from V.

See also Appendix F for tools regarding Step 3 of ant travels without using a little finger.

Part 2 - Building a Solution - page 7 - Entire Construction

Part 3 – Q.E.D.

The presentation and additional material in appendices demonstrate how to trisect an angle using only Euclidean methods and tools. The ants travel along an arc. Ant A always travels twice the distance as Ant B.

If repeated and controlled practice, or observation, contradicts theory then the theory is generally changed.

"According to all known laws of aviation, there is no way that a bee should be able to fly. Its wings are too small to get its fat little body off the ground. The bee, of course, flies anyway. Because bees don't care what humans think is impossible."[36]

Quod erat demonstrandum.

Means
which was to be demonstrated,

but literally translates to
what was to be shown.

This demonstrates a method to
trisect an angle within the limits
and with the tools available at the time.

Part 3 page 1 - QED

As theory this method may never be "proven". In practice it appears to have been "demonstrated".

As Galileo[37] might have said to the media when exiting the court:

"yet it still works."

Appendices

These are not necessarily referenced in this order.

Appendix A – Line Trisection

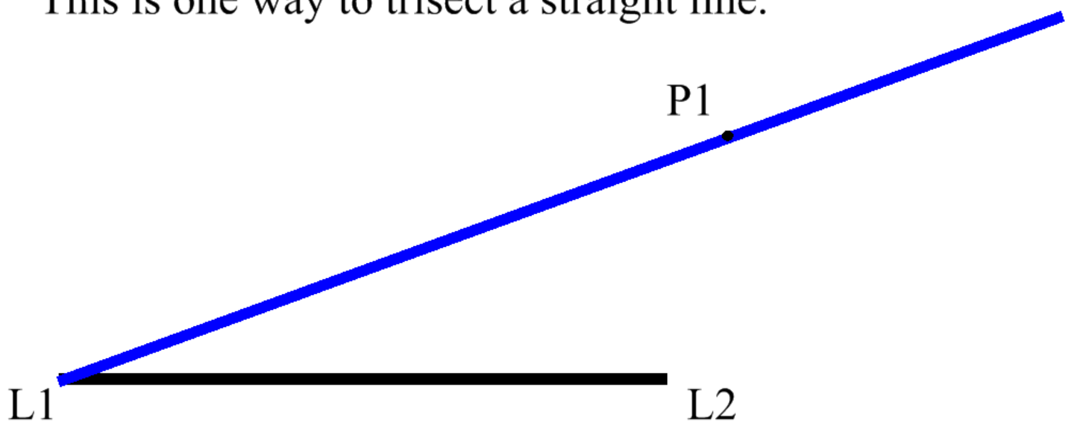
Title

Appendix A

Trisecting A Line

Pages 1 through 4

This is one way to trisect a straight line.

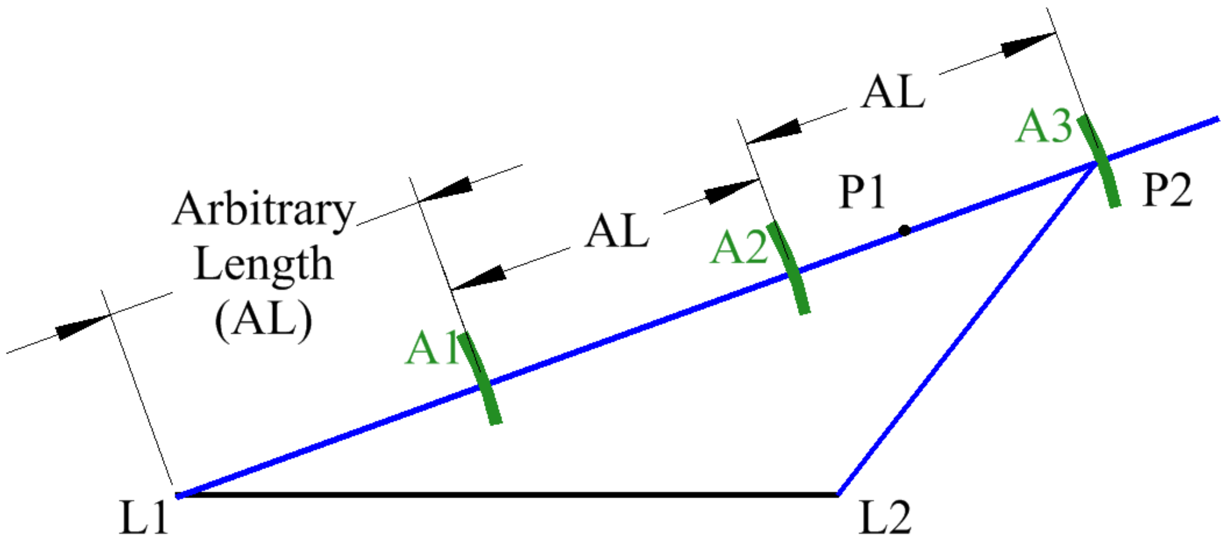


L1-L2 is a line.

Create Point 1 (P1) up and to the right of L2.

Draw a line from L1 to P1 and beyond.

Appendix A - Line Trisection - page 1

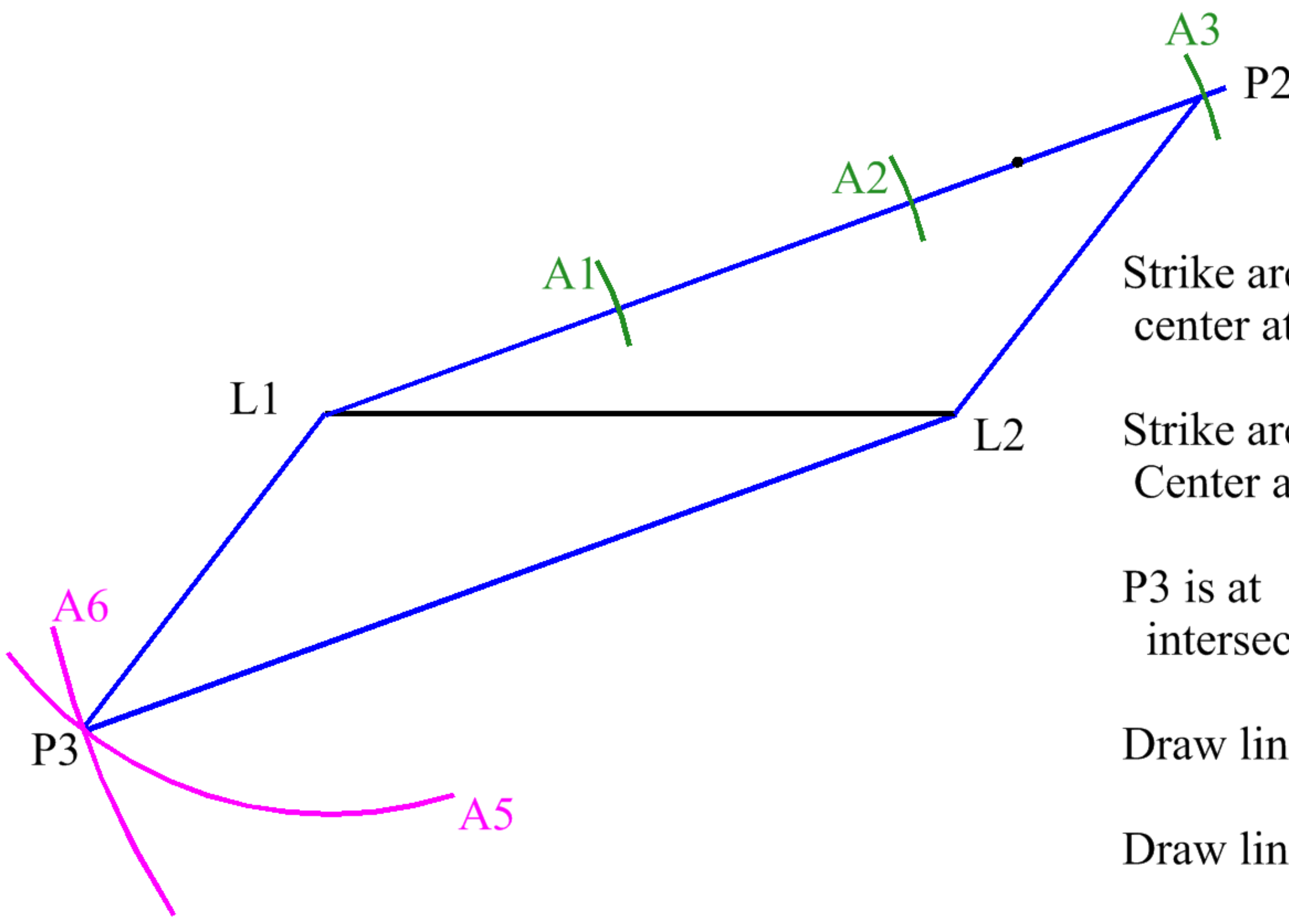


Strike arc A1 center at L1 and arbitrary radius.
Create arcs A2 and A3 so A3 is further away
from L1 than P1 is.

P2 is where the arc A3 crosses the extension of L1-P1.

Draw line L2-P2.

Appendix A - Line Trisection - page 2



Strike arc A5
center at L1 radius equal to L2-P2

Strike arc A6
Center at L2 radius equal to P2-L1

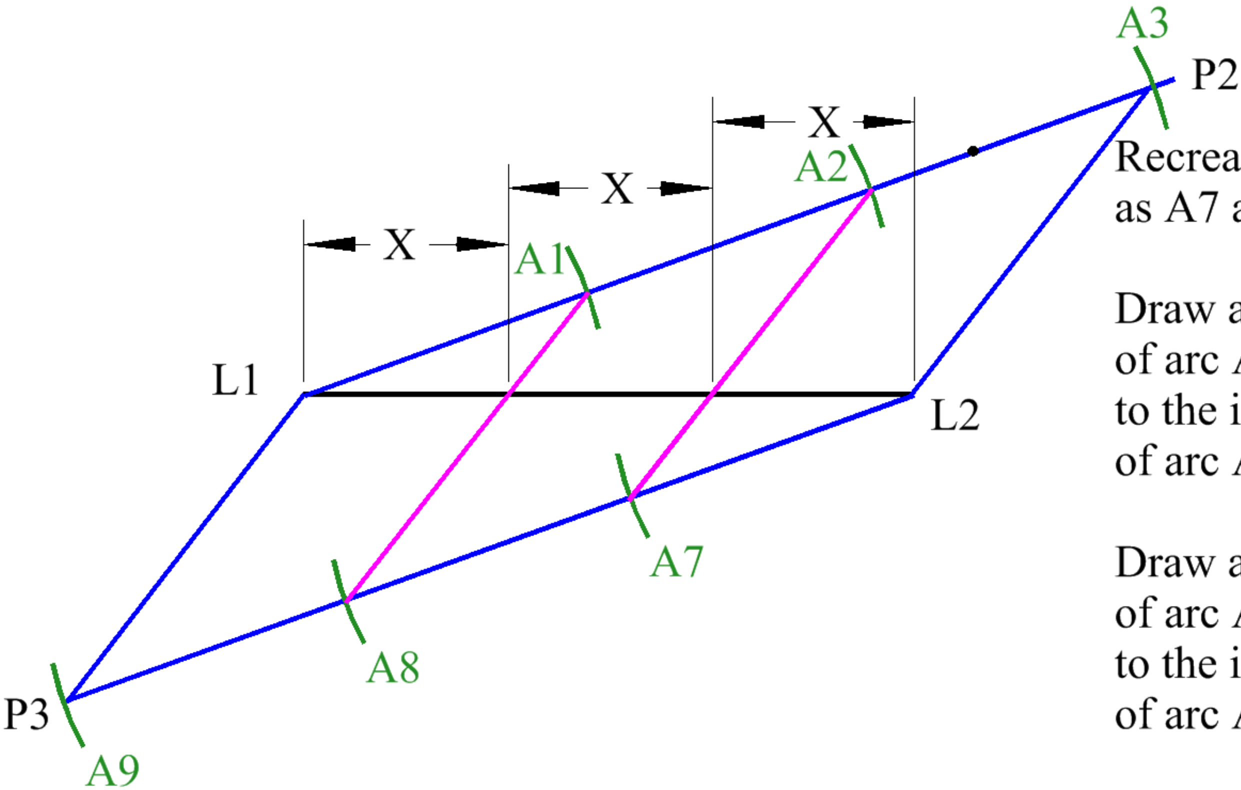
P3 is at
intersection of A5 and A6

Draw line from L1 to P3

Draw line from L2 to P3

arc A4 not used

Appendix A - Line Trisection - page 3



Recreate the arcs A1 and A2
as A7 and A8 along L2-P3

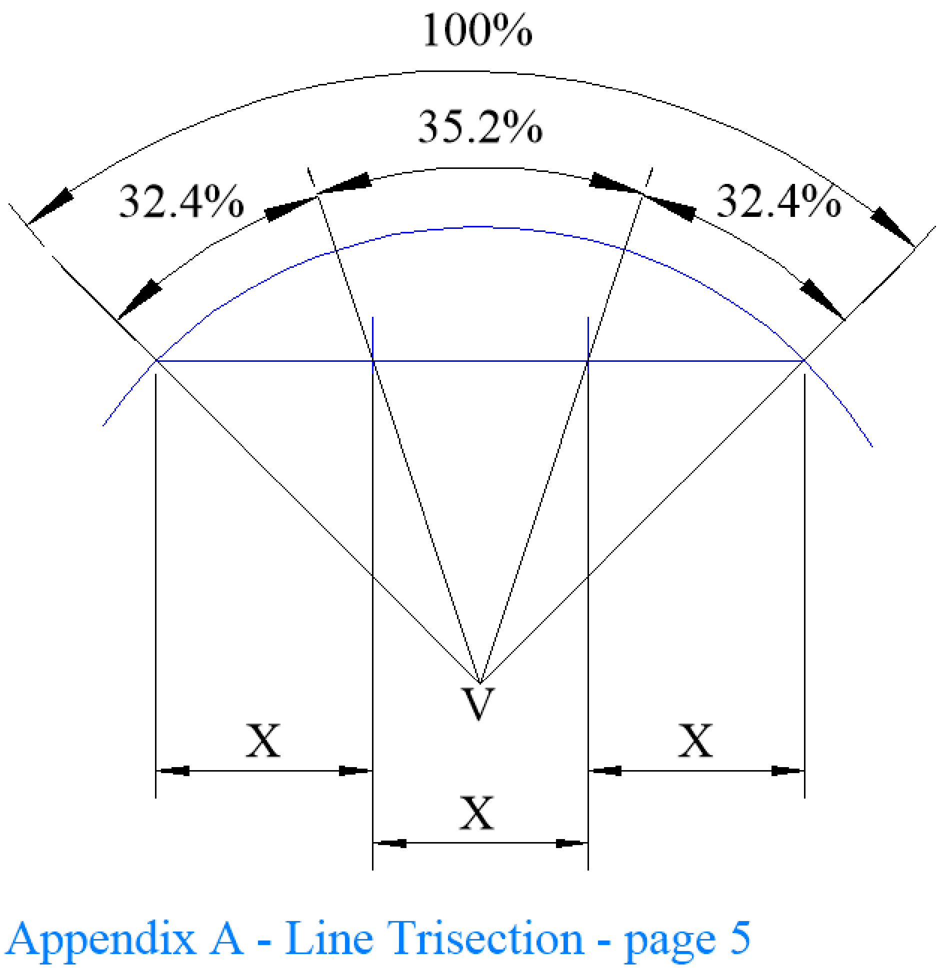
Draw a line from the intersection
of arc A1 and line L1-P2
to the intersection
of arc A8 and line L2-P3

Draw a line from the intersection
of arc A2 and line L1-P2
to the intersection
of arc A7 and line L2-P3

Line L1-L2 has been trisected into three equal lengths where each section is X long. Len(X+X+X) = Len(L1 to L2).

Appendix A - Line Trisection - page 4

Page 5 – Line Yes. Arc No.



So why won't this work for arcs?

While the chord under an arc is a straight line the arc itself is not.

The chord was trisected as shown earlier. The rays run from V through the chord to the arc.

The left most and right most angles are identical due to the symmetry about a vertical from V up.

The center third is a wider angle.

The percentages shown apply only to this example.

Appendix B – Infinitesimal Geometry

Title

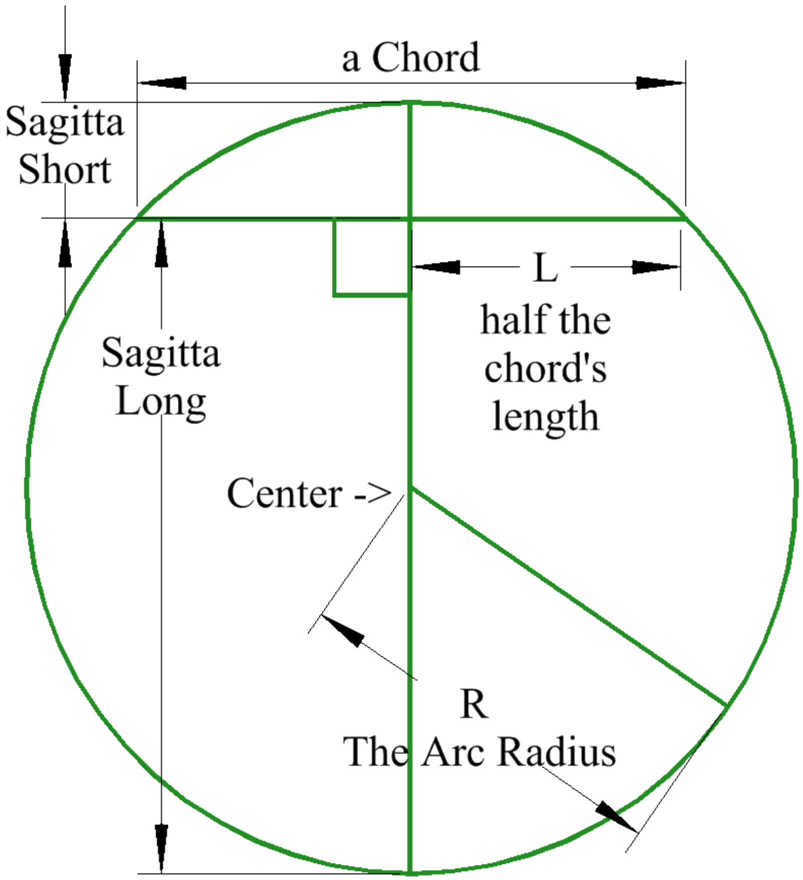
Be sure to read page 12 of this appendix.

Appendix B

Infinitesimal Geometry

Page 1 – Sagitta

Infinitesimal is known, but outside of observation capability.



A chord is a straight line connecting to the arc at both ends and whose perpendicular bisector goes through the center.

The vertical distance perpendicular (here up or down) from the chord's center to the arc is a “Sagitta”.

The sagitta can be calculated from the lengths of the radius and chord without referencing an angle.

“S-” is also called “Sagitta-Short or S-Short”
= R - SQRT(R^2 – L^2)

The long sagitta runs perpendicular from chord through center to other side of arc.

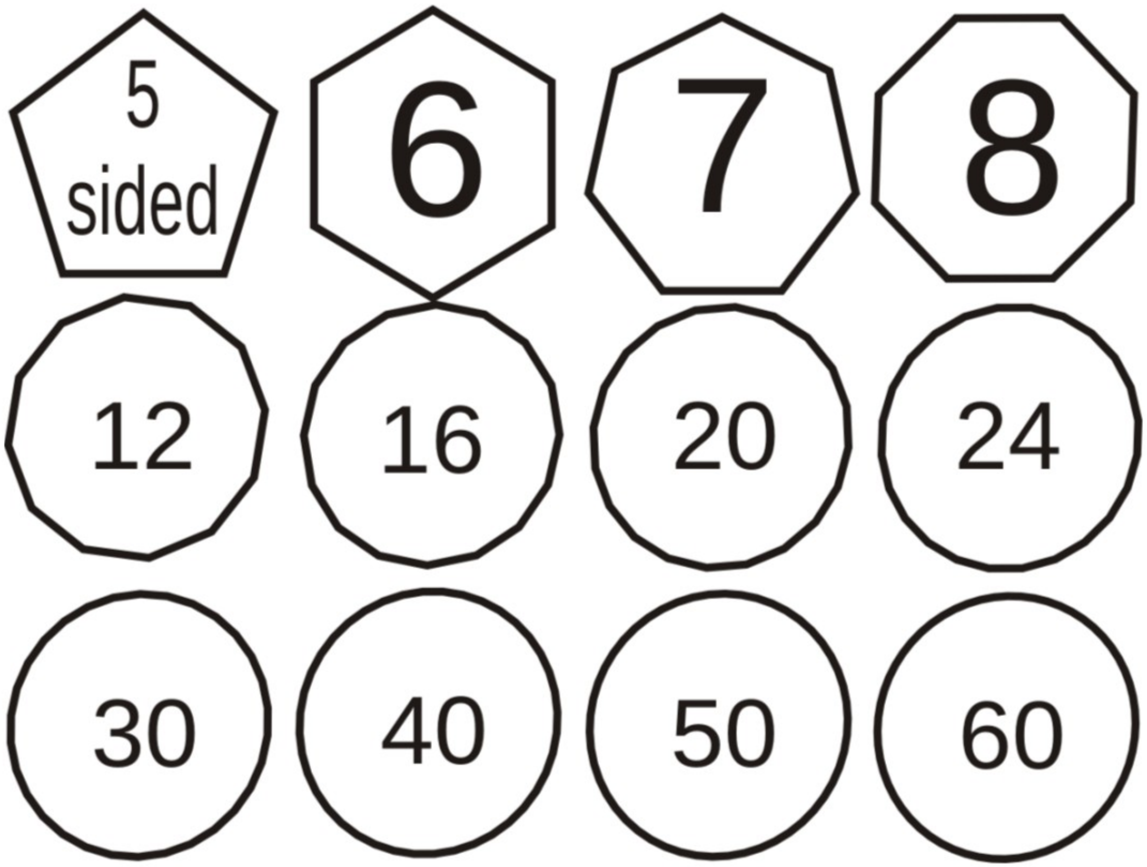
“S+” is also called “Sagitta-Long or S-Long”
= R + SQRT(R^2 – L^2)

Appendix B - Infinitesimal Geometry - page 1

In (S+) + (S-) the two square roots cancel out so the sum of both sagitta is 2R, a diameter, as indicated in the drawing.

Page 2 – 60-sided circle

The questions are:
At what point is the arc segment above the chord indistinguishable from that chord?
Is there a place where a difference makes no difference?
Or that difference can not be observed?



Here are a dozen shapes.

At what point can you not see each increasingly short line of their composition?

The last one has 60 sections. Each is about 0.01066" long or about 0.2709 millimeters.

The perception is affected by line thickness. The thinner the line the more you can "see" the transition (inflection) from one segment to the next.

Appendix B - Infinitesimal Geometry - page 2

Page 3 – Two Examples

The questions are:
At what point is the arc segment above the chord indistinguishable from that chord?
Is there a place where a difference makes no difference?
Or that difference can not be observed?

Consider a huge example:

Radius =1 mile (inches)	Length of Chord C (inches)	Length of L (inches)	Sagitta Short (inches)	Sagitta Short (mm)
63360	1	0.50	0.00000197	0.000050

five hundred-thousandths of a millimeter

Consider a large example:

Radius 5 yards (inches)	Length of Chord C (inches)	Length of L (inches)	Sagitta Short (inches)	Sagitta Short (mm)
180	1	0.50	.00069	0.017639

about the diameter of a thin human hair
and much less than the diameter of a 0.5 mm pencil lead.

Appendix B - Infinitesimal Geometry - page 3

Human hair diameter – see endnote 41 on page 96.

Consider a chord, short in relationship to the radius of the arc.

The rules preclude a measurement device, but we have Rules of Thumb, Finger and Hand.

Adult spread fingers range from about 5” to about 8” measured from tip of thumb to tip of little finger.

Width of adult little finger at the last pad generally ranges between 3/8” (0.375”) and 5/8” (0.625”).

So at any radius 4” to 7” with a chord as long as a little finger is wide (even at 0.625”) gets a sagitta-short of under 0.5 mm. (see table)

If the sagitta-short is under 0.5 mm using a line width of 0.5 mm merges the lines for the chord and arc merge to an infinitesimal difference.

0.5 mm is a fine mechanical pencil, not available to Euclid or the Babylonians.

The table on the next page was created to show the sagitta-short for varying radii and chord length. The values in the table are __not__ part of how to create the trisection.

If using a finger conflicts with your understanding of “the rules” see Appendix F.

Appendix B − Infinitesimal Geometry − page 4

Page 5 – Some Sagittas

Here is a table of arc radii from 3 to 7 inches, chords 1" or less and the resulting sagitta-short in millimeters. S-Short under 0.5mm are bold.

The longer the radius the smaller the S-Short. The smaller the chord the smaller the S-Short

The 0.5" chord is used in the table as something close to the width of the pad between the second and third joint of the little finger.

Using that gets a S-Short much smaller than 0.50mm for a convenient radius >= 4.0” to a radius <=7.0”.

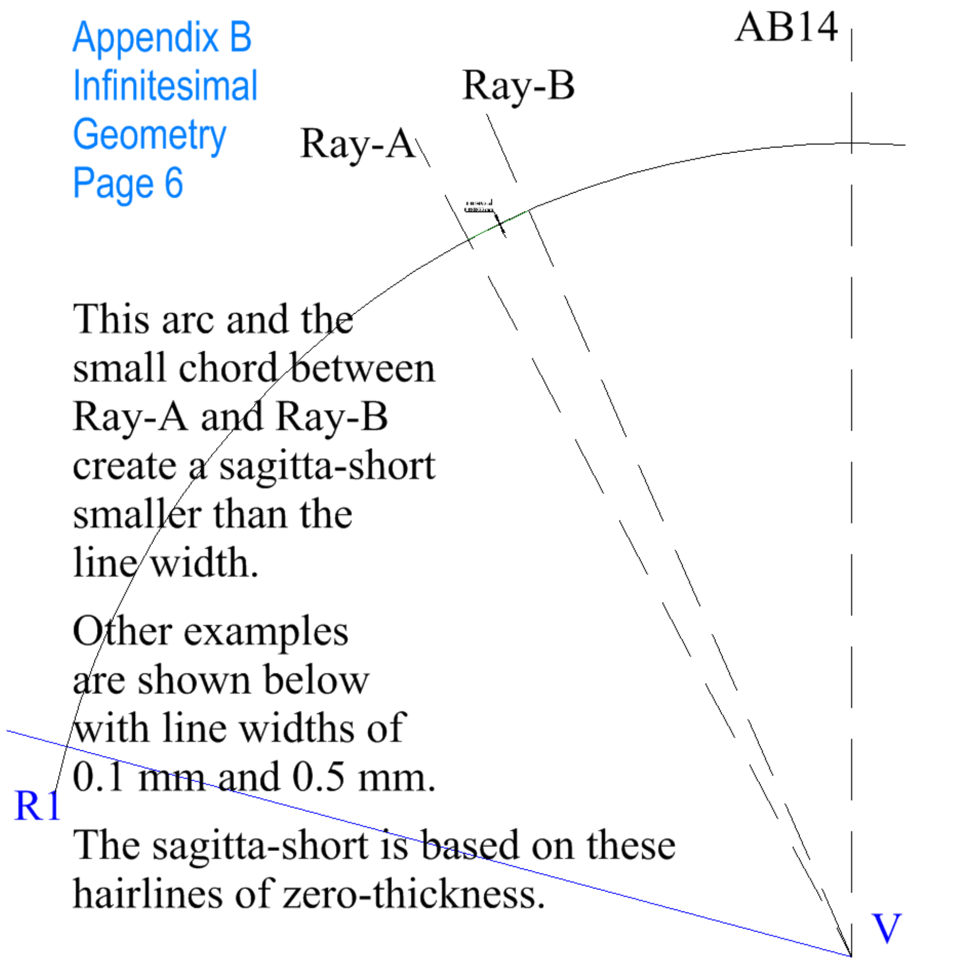
A radius larger than 7" may not fit the single sheet of 8.5” x 11” paper.

Appendix B - Infinitesimal Geometry - page 5

Radius of Arc R (inches)	Length of Chord C (inches)	L (inches)	S-Short (inches)	S-Short (mm)
3	1.000000	0.500000	0.041960	1.065787
3	0.680000	0.340000	0.019329	**0.490955**
3	0.500000	0.250000	0.010435	**0.265044**
4	1.000000	0.500000	0.031373	0.796875
4	0.750000	0.375000	0.017617	**0.447470**
4	0.500000	0.250000	0.007820	**0.198632**
5	1.000000	0.500000	0.025063	0.636595
5	0.850000	0.425000	0.018095	**0.459619**
5	0.500000	0.250000	0.006254	**0.158849**
6	1.000000	0.500000	0.020870	0.530089
6	0.950000	0.475000	0.018832	**0.478324**
6	0.500000	0.250000	0.005211	**0.132349**
7	1.050000	0.525000	0.019715	0.500768
7	1.000000	0.500000	0.017880	**0.454151**
7	0.500000	0.250000	0.004466	**0.113429**

Page 6 – Line Width

This image is shown at zero magnification, that is 100% or 1x.

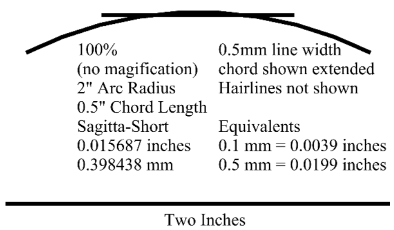


5 inch line for scaling

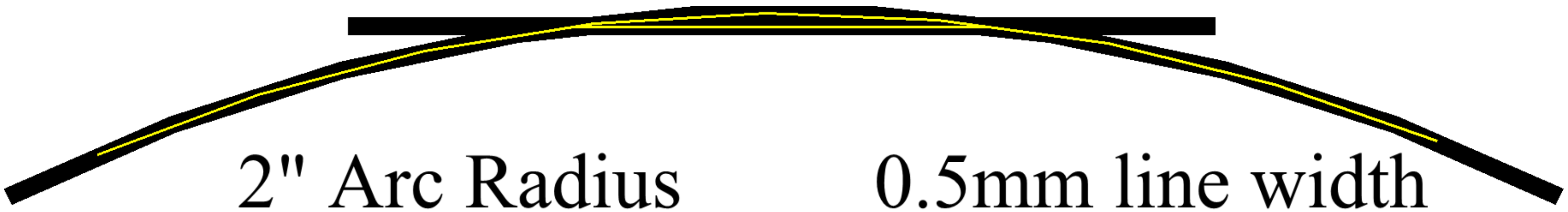
Page 7 – 0.5 mm at 1:1

How line width affects observational ability in reality.

Can you see the sagitta-short?



Page 8 – 0.5 mm Magnified
How line width affects observational ability in reality.



2" Arc Radius
0.5" Chord Length
Sagitta-Short
0.015687 inches
0.398438 mm

0.5mm line width
chord shown extended
Hairlines in yellow

Equivalents
0.1 mm = 0.0039 inches
0.5 mm = 0.0199 inches

Two Inches for Scale

Page 9 – General Rule

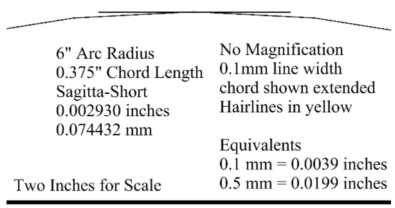
Half a line width to draw the chord will appear in the area between the chord and the arc. Half the line width to draw the arc will appear in the area between the arc and the chord. Hence:

If the sagitta-short is less than the line width then the sagitta-short will be obscured, not observable.

See a graphic on Appendix B – Page 12. Paraphrasing Wantzel, Bhat wrote: "***It is <u>not</u> possible to trisect an angle of any given value, <u>with the tools and knowledge available at that instant of time.</u>"*** (see endnote 40 on page 95) Did Euclid have a magnifying glass[38]? If so, would that be a "tool" hence disqualifying?

Page 10 – 0.1 mm at 1:1

Line widths affects observational ability. Can you see the sagitta-short in the image to the right? It is at 1x, not magnified. The sagitta-short is less than 0.1 mm, the finest (thinnest) line readily available (see endnote 39 on page 95) and Euclid didn't have it.



Page 11 – 0.1 mm Magnified

Magnified at over 8000%

This is an 80^{+}x magnification of the same sagitta-short as shown at 1x above. The yellow hairlines show the sagitta-short, but it is obscured in reality. It is known, obscured, not observable, hence infinitesimal.

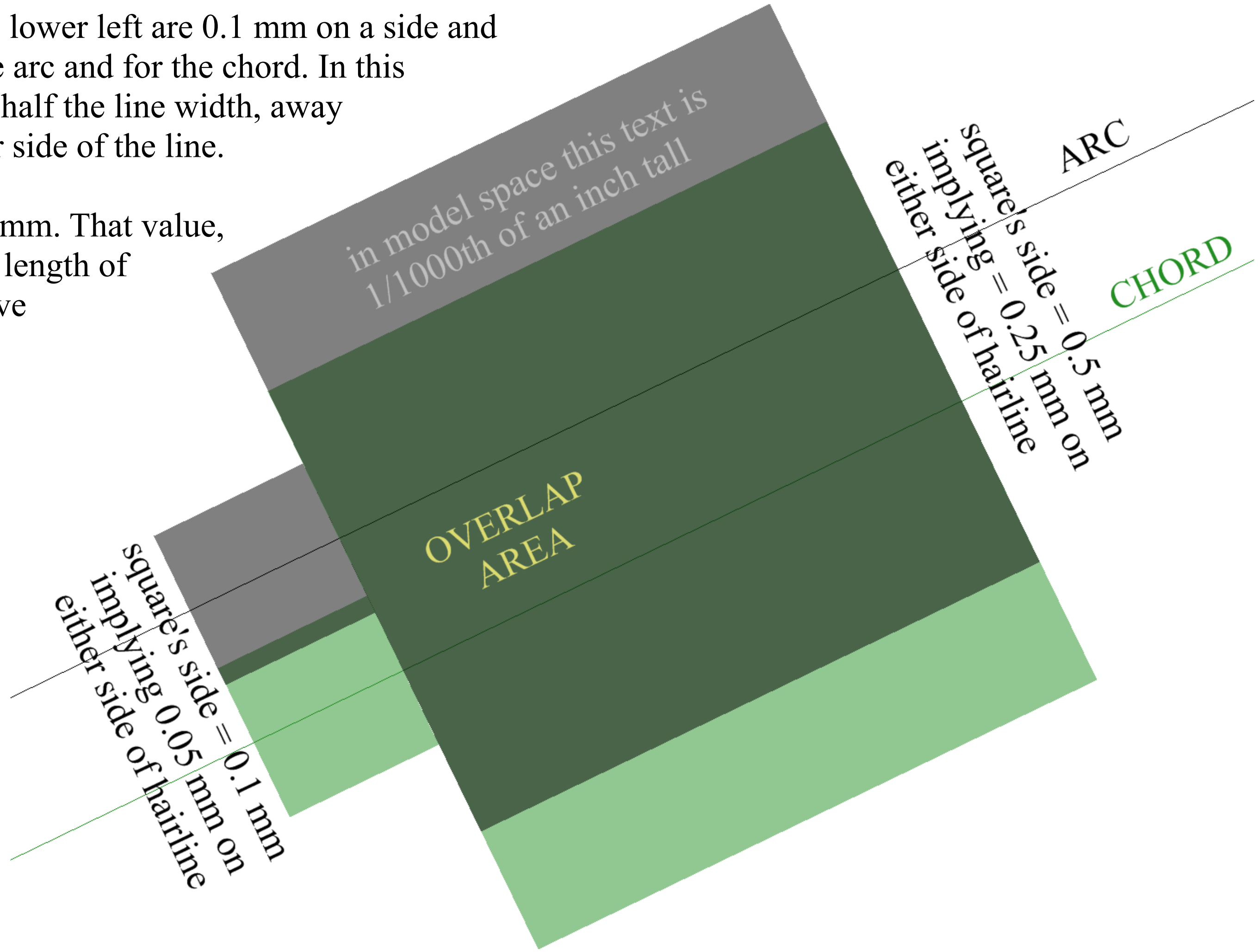
Page 12 – Sizing & Implications

The two smaller squares at the lower left are 0.1 mm on a side and centered on the hairline for the arc and for the chord. In this manner they extend 0.05 mm, half the line width, away from the arc or chord on either side of the line.

This sagitta-short is 0.088822 mm. That value, less 2 x 0.05mm (half the side length of each small square), is a negative number shown as an overlap.

This means the sagitta-short is obscured when the chord and the arc are drawn with a 0.1 mm line width.

The two larger squares at the upper right are similar constructions using a 0.5 mm line width drawn centered on the hairline of the arc and the chord that would obscure the sagitta as indicated by the label “overlap area”.



Where the two pairs of squares meet is the sagitta-short, the widest separation of this chord and this arc.

See summary on next page.

Summary of Implications

So: an arc and chord with sagitta-short smaller than the line width may be trisected via rays through the trisected chord with an infinitesimal effect.

There is no observable difference using a 0.1 mm line width available in the finest (thinnest) mechanical pencil reportedly available[39] circa 2024.

The more common 0.5 mm mechanical pencil obscures sagitta-shorts up to 0.5 mm.

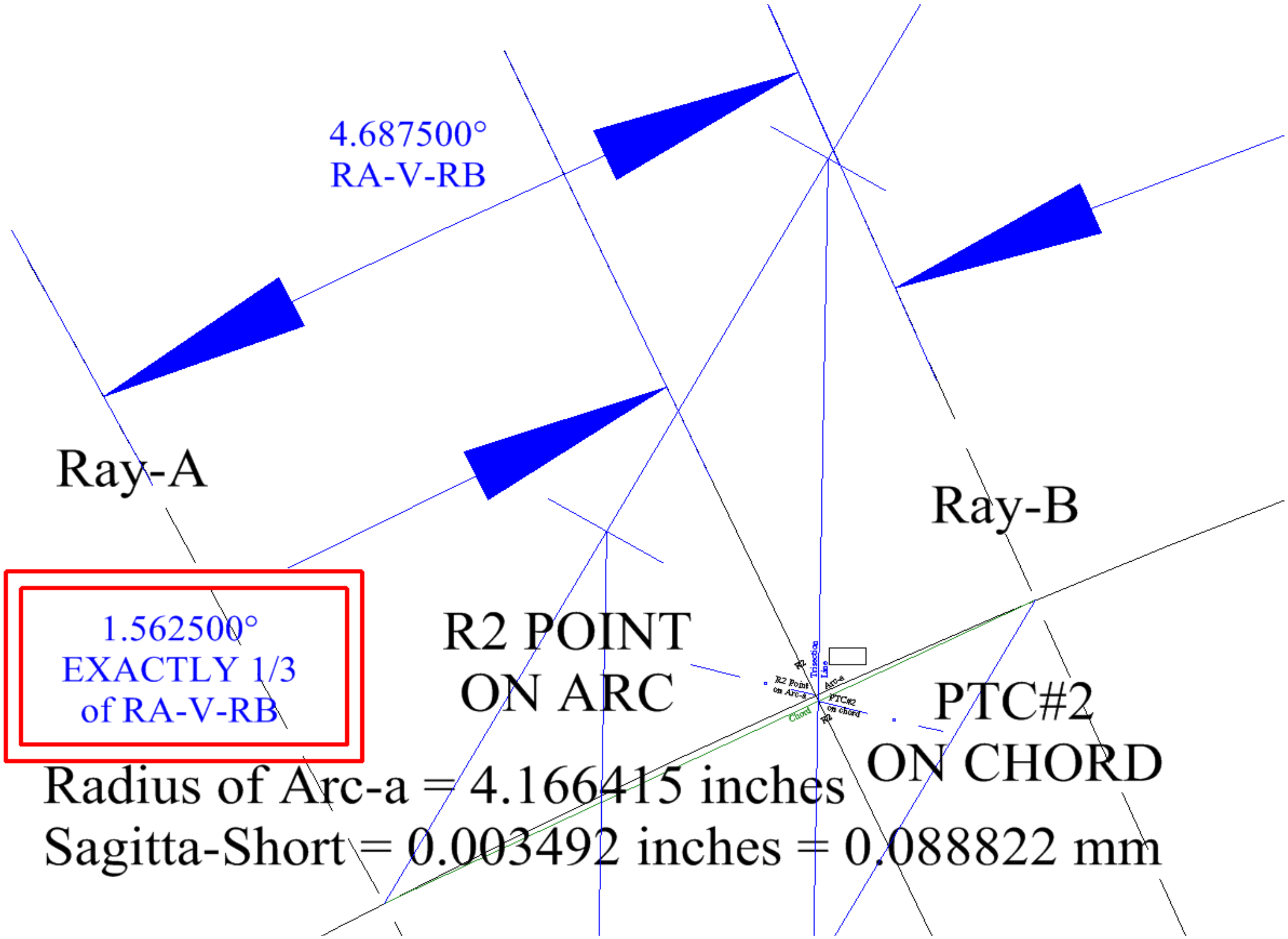
While not well known, it seems reasonable that Euclid didn't have anything that wrote or drew with a finer (thinner) line.

Users of the earliest reed pens per *Presentation | Writing Tools | Reed & Quill* on page 5 generally had wider lines which would obscure an even larger sagitta-short. As indicated by the table shown on *Appendix B*, page 5 small sagittas occur more frequently with longer rays and shorter chords.

An optional technique that provides more paper space to draw the chord trisection is presented in Appendix G starting on page 68.

Is this technique accurate? See pages Appendix B, Page 13 and Appendix B, page 14 below.

Page 13 – How Accurate?



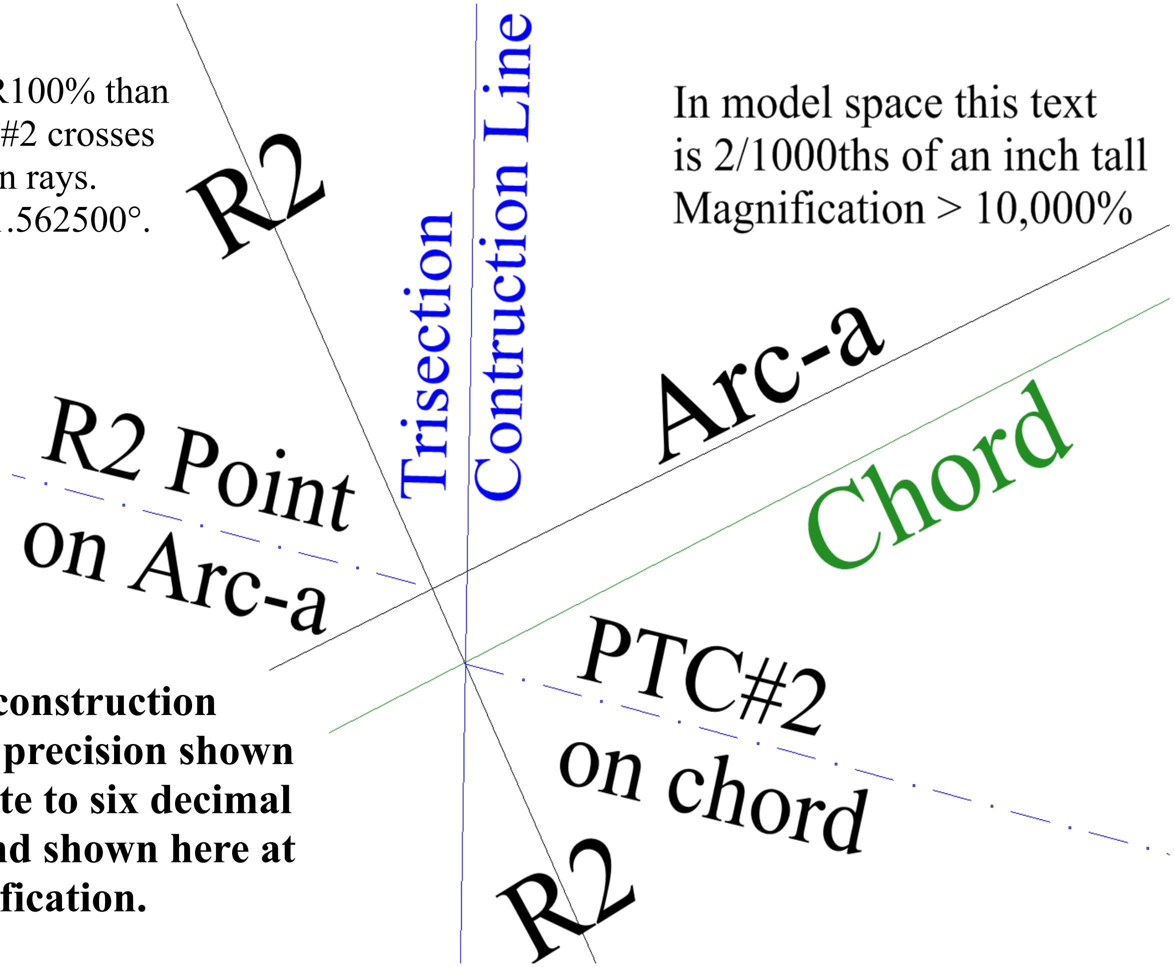
This example shows Ray-A and Ray-B from [Appendix B - page 6](). The trisection was accurate. The blue lines are part of the chord trisection construction or related dimension measurements.

Page 14 – How Accurate (Magnified)

Chord trisection gets us PTC#2, closer to R100% than PTC#1. Where the extended line of V-PTC#2 crosses Arc-a is a point on R2, one of two trisection rays. RA-V-RB measures 4.687500°. A third is 1.562500°.

Paraphrasing Wantzel, Bhat wrote: It is not possible to trisect an angle of any given value, with the tools and knowledge available at that instant of time.[40]

Using only those tools, this example demonstrates a trisection construction respecting the angle and creating a precision shown with angular measurements accurate to six decimal points, one millionth of a degree, and shown here at more than a 10,000% (100x) magnification.



This the last page of *Appendix B* describing *Infinitesimal Geometry* and the implications of a sagitta-short smaller than the line width of the drawing tool.

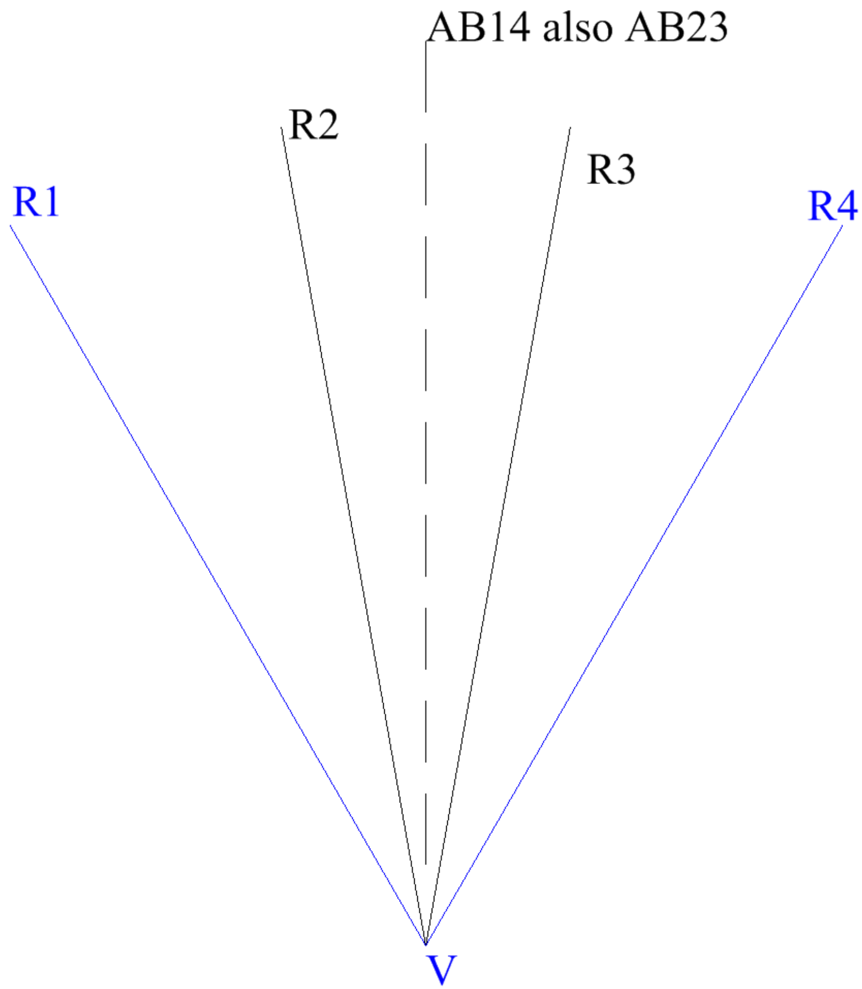
The best place to start is the Introduction on page 4. To better understand the construction see the Presentation starting on page 7. More examples under different conditions and some options appear in the following appendices.

Appendix C

Range Narrowing

To get to a short chord we need to narrow the range of where we expect R2. We started with AB14, determinable from the givens and allows us to work with half the situation. R3 is a mirror of R2 on the other side of AB14.

Page 1 – Ants & Clocks



Appendix C - Range Narrowing - page 1 - Ants & Clocks

Recall from "R2 at the solution" Angle R1-V-R2 is twice as large as angle R2-V-AB14.

Consider an arc with center V and running from R1 to AB14 at any radius.

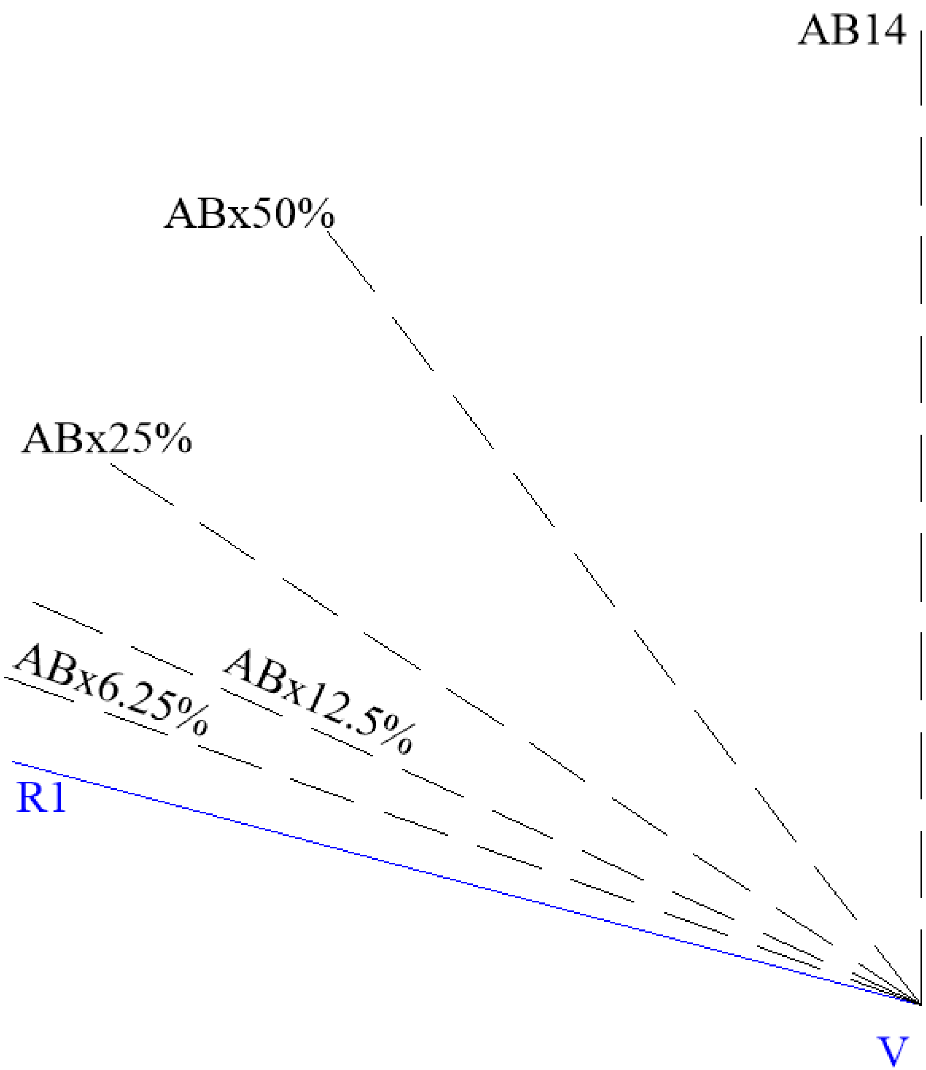
Ant-B started walking along the arc starting at AB14 and moving toward R1 at speed X.

Ant-A started walking along the same arc, but started at R1 and walked toward AB14 at speed 2X

The two ants would meet at R2.

Clock analogy: A hand starting at R1 will have to run twice as fast clockwise to reach another hand at AB14 running counterclockwise to meet at R2.

Page 2 – Building Tools



AB14 is the angle bisector of R1-V-R4.

We're only going to work with half the given angle. Label your ABs!

R2 runs at 2/3 of the R1-V-AB14 angle. Or 1/3 of the whole R1-V-R4 angle.

0% is at R1. 100% is AB14.

ABx50% is the bisector of R1-V-AB14.
ABx25% is the bisector of R1-V-ABx50%
ABx12.5% is the bisector of R1-V-ABx25%

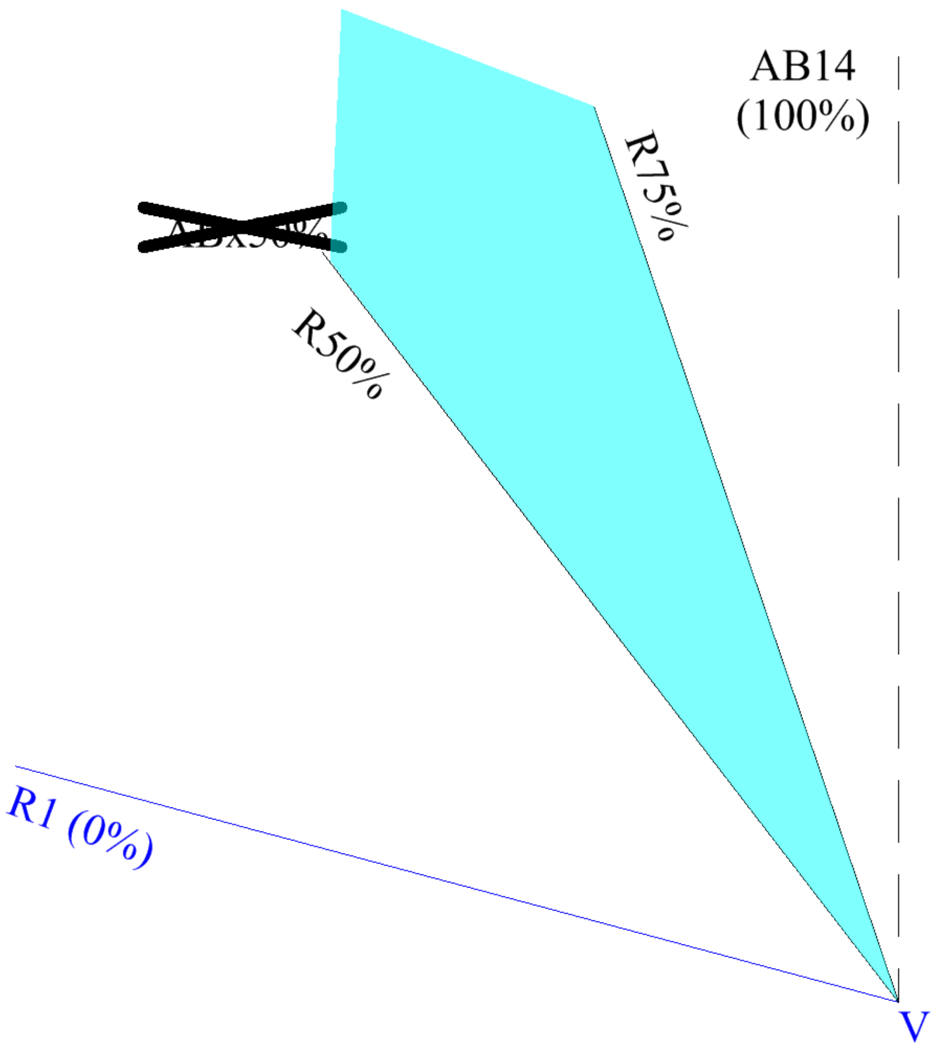
ABx6.25% is the bisector of R1-V-ABx12.5%

These are building tools.

ABx50% is NOT 50-degrees.

Appendix C - Range Narrowing - page 2 - building tools

Page 3 – Range Narrowing #1



Copy the angle R1-V-ABx25% to AB14-V-R75%. That is 25% counterclockwise from AB14.

R is for RAY because this is not an angle bisector at this location. 75% is because it is 25% less than the 100% at AB14.

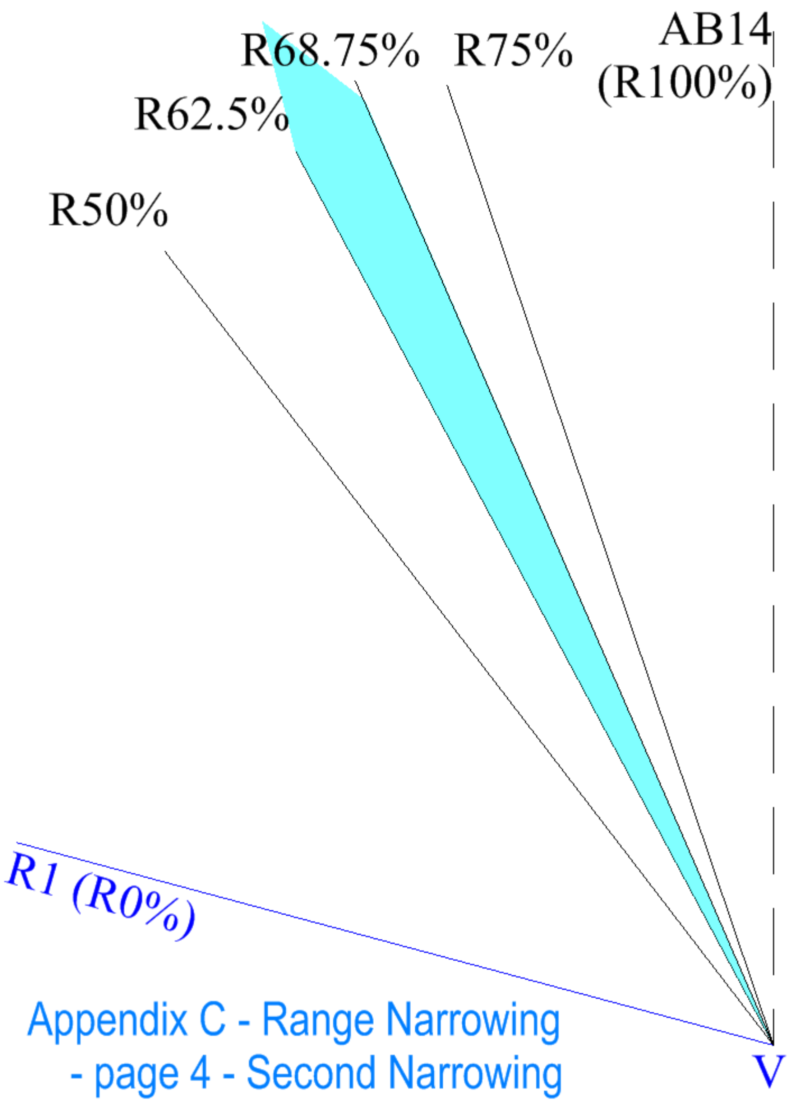
Note that R50% is the new name for ABx50%. Run your pencil over the dashed line for a solid line. (if you want to)

R50% is short notation for "Ray at 50% of the arc from R1 towards AB14" Similar for all Rnn% labels

We've narrowed the zone for R2 to the light blue area shown. The "zone of solution" extends away from the vertex.

Appendix C - Range Narrowing - page 3 - First Narrowing

Page 4 – Range Narrowing #2



The tools in the area COUNTER CLOCKWISE from R50% are outside the zone of solution. For clarity the tools are not shown here.

Copy angle R0%-V-ABx12.5% CLOCKWISE of R50%.

Label this line R62.5% (50%+12.5%)

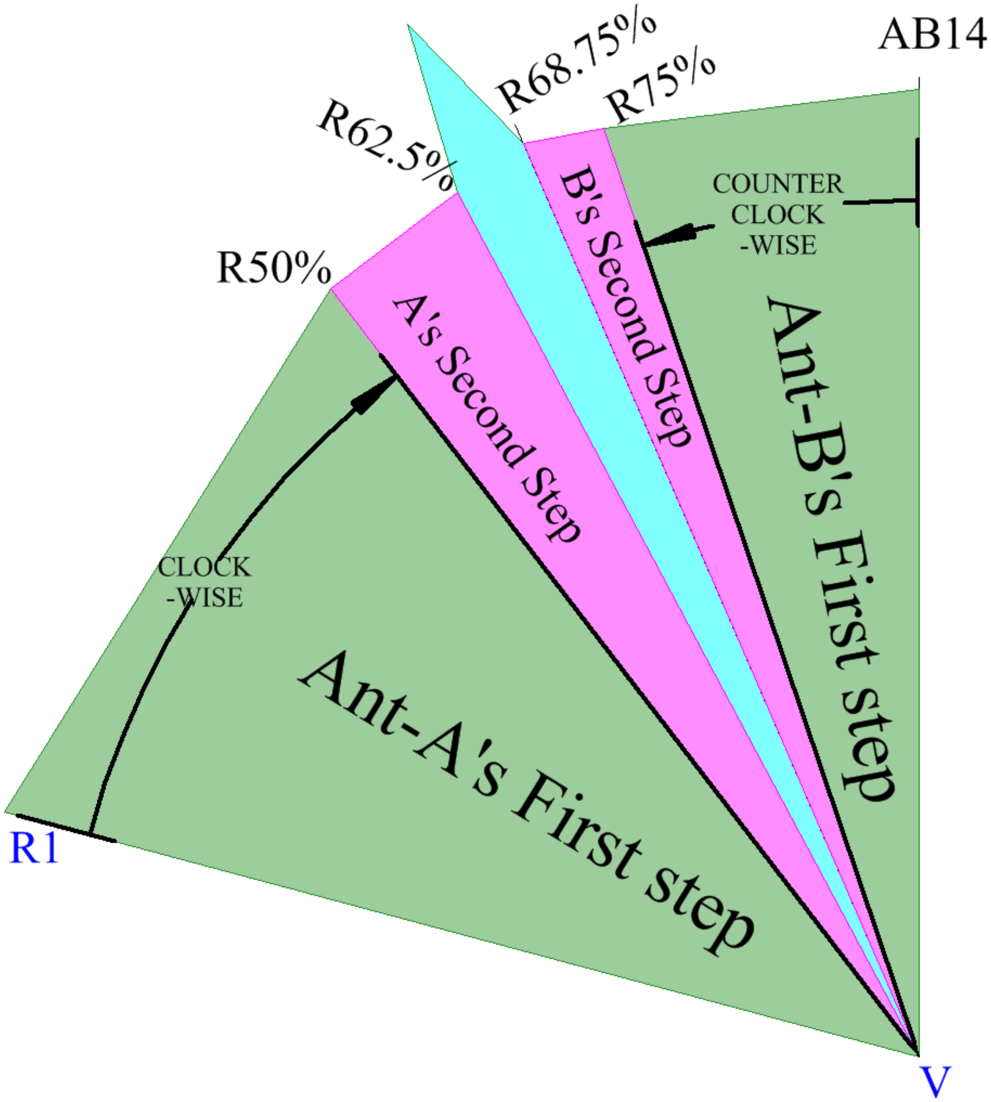
Copy the angle R0%-V-ABx6.25% COUNTER CLOCKWISE of R75%. Label this line R68.75% (75%-6.25%)

The zone of solution has been narrowed a second time.

Both times the CLOCKWISE addition has been TWICE angle of COUNTER CLOCKWISE angles.

Ant-A is still moving twice as fast as Ant-B.

Page 5 – Step Review



Appendix C - Range Narrowing - page 5 - Step Review

Ant-A moved from R1 along any arc towards AB14.

Ant-B moved from AB14 along the same arc and towards R1.

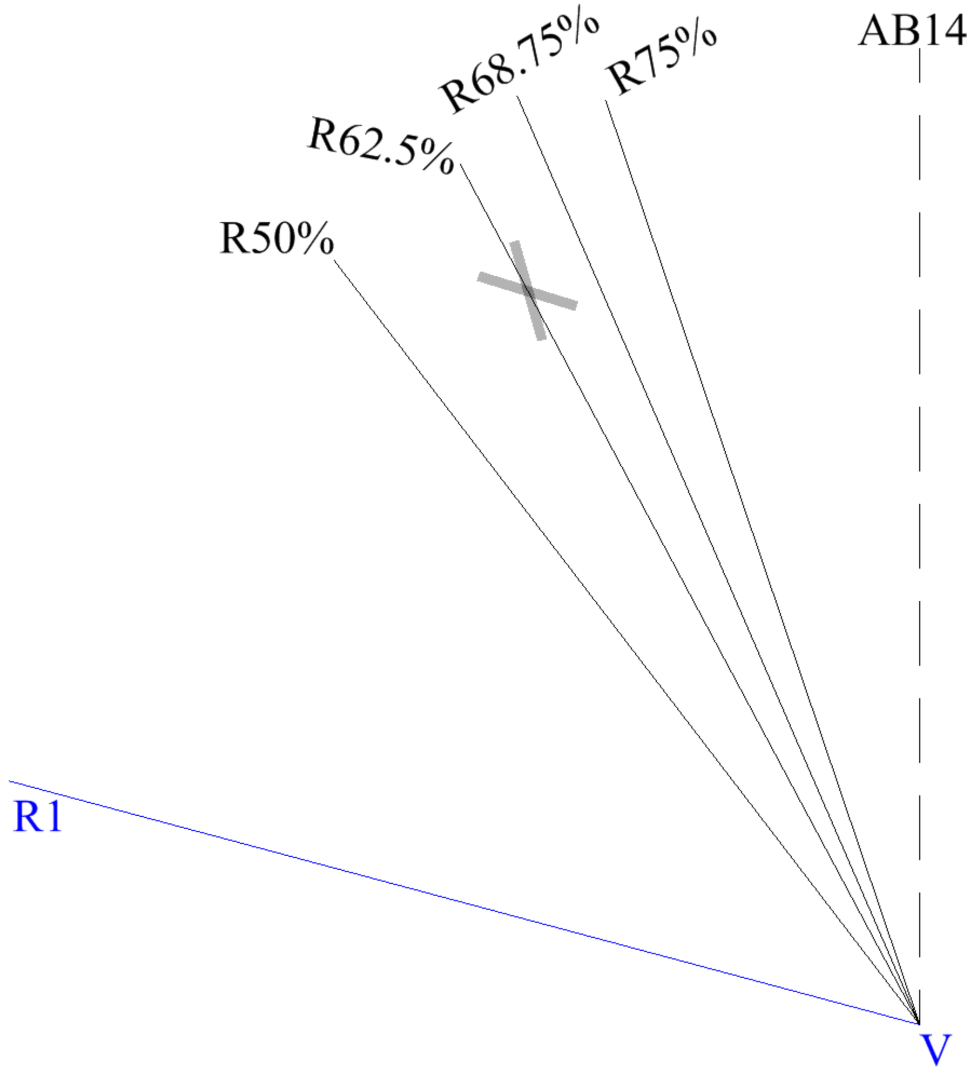
Ant-A moved twice as fast as Ant-B.

First step for each ant is shown in green.
Second for each is shown in magenta.

The light blue is where R2 resides.
It is also called the "Zone of Solution" and, like rays and bisectors, it extends from Vertex V to infinity.

We could do this more times, but it isn't necessary. The purpose of these two steps is complete. Now to draw an arc, but where?

Page 6 – Giving A Finger



Next put your little finger's last pad between R68.75% and R62.5%.

If your pad is wider than those two rays move your finger away from the vertex until it just fits.

If your finger is narrower than those two rays move your finger toward the vertex until it just fits.

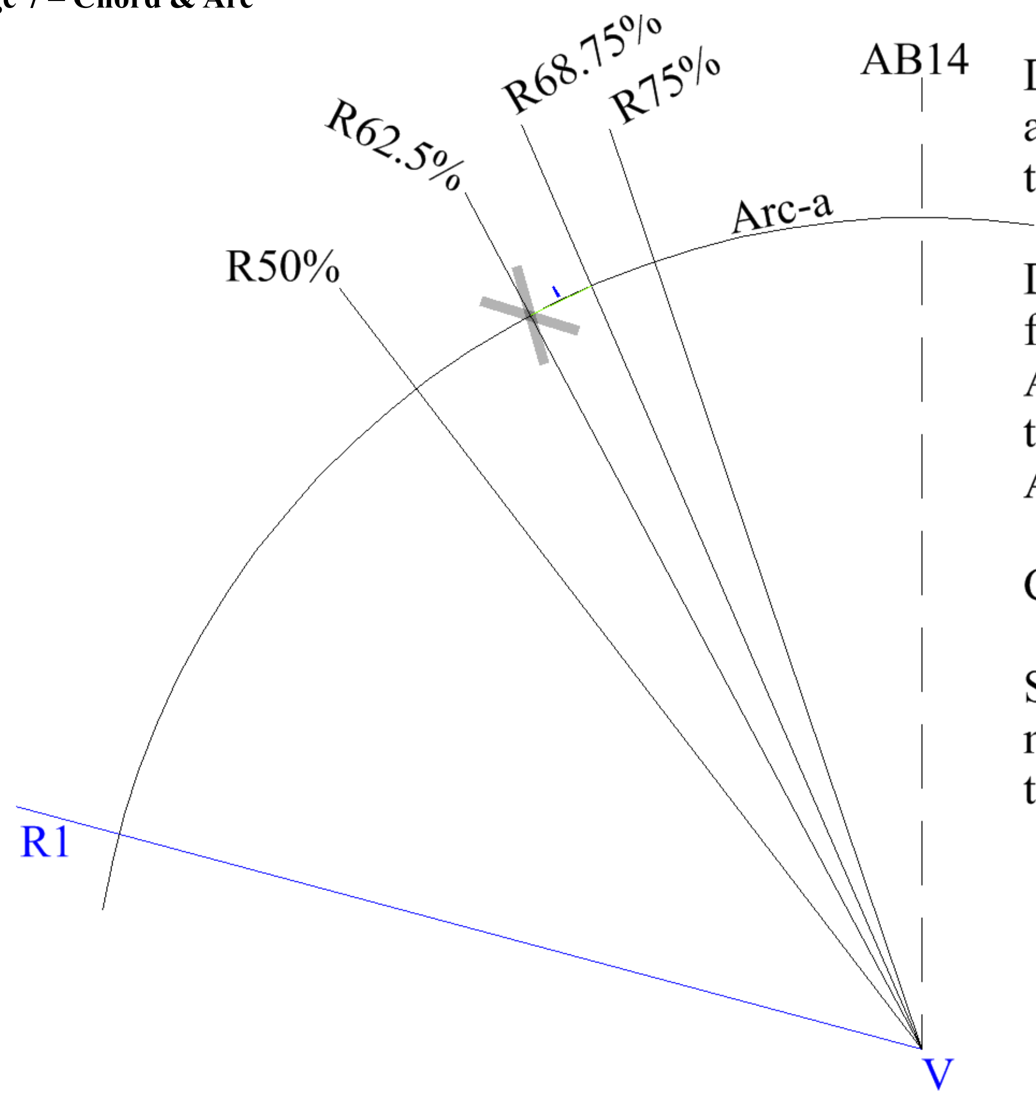
Make a mark on either R62.5% or R68.75% where your pad rests.

Don't use the tip of your finger. Use the pad, a little closer to your wrist than the tip, about halfway from the last finger joint to the tip.

I marked mine with a partially transparent X on R62.5%.

Appendix C - Range Narrowing - page 6 - Giving the problem your finger

Page 7 – Chord & Arc



Draw an arc with the center at V and the radius to where X marks the spot. Label it "Arc-a".

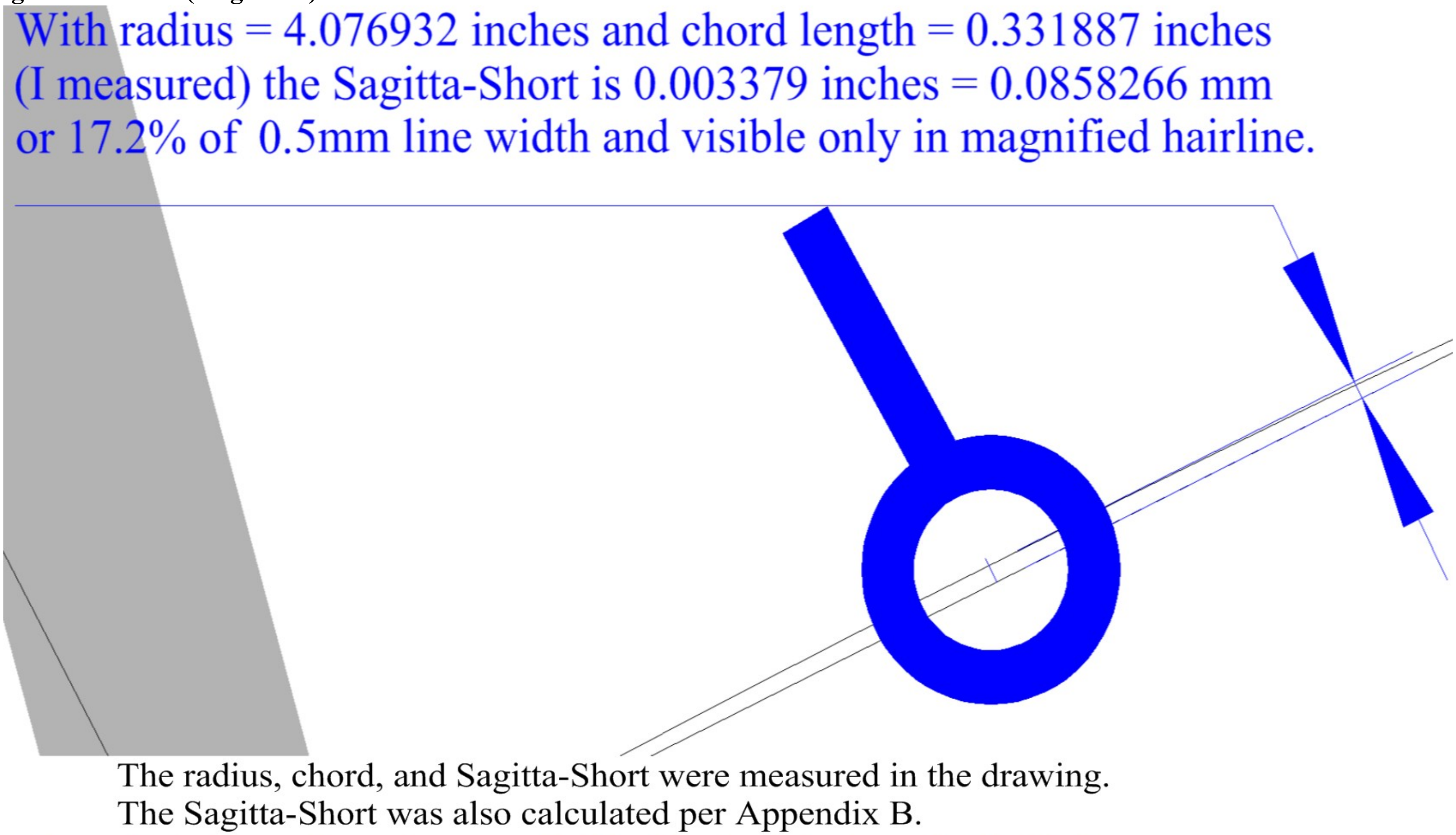
Draw a chord
from the intersection of
Arc-a and R62.5%
to the intersection of
Arc-a and R68.75%.

Can you see it?

See next page for a magnified view of the tiny blue writing.

Appendix C - Range Narrowing - page 7 - The Chord and Arc

Page 8 – Hairline (magnified)



Appendix C - Range Narrowing - page 8 - The Chord and Arc (magnified hairline)

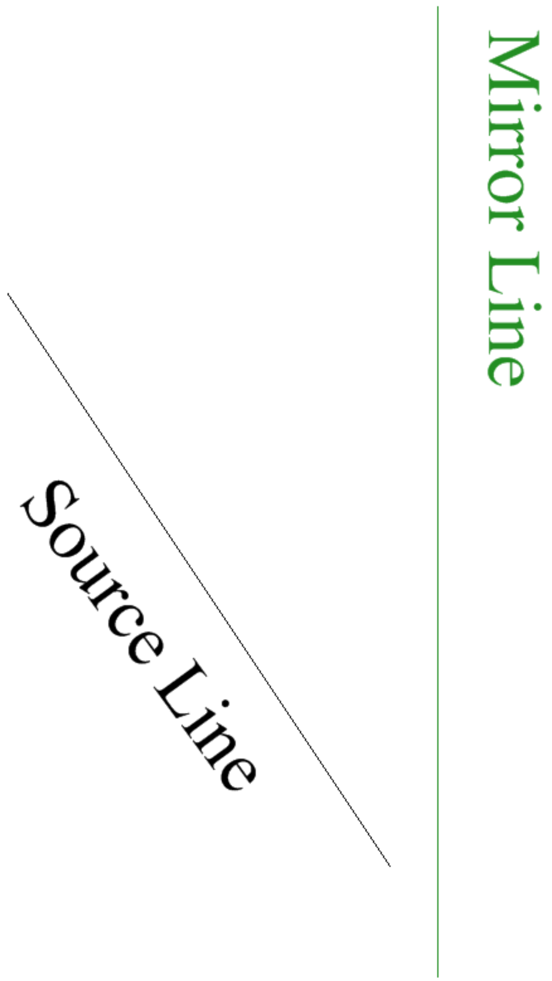
The sagitta-short, being less than 0.1 mm, will be obscured by half the line width of a 0.1 mm line centered on the arc and half the line width of a 0.1 mm line centered on chord. See *Appendix B* starting on page 25 for magnified views. Don't like the idea of using a finger pad? See Appendix F on page 59 for a third pair of tools.

Appendix D

Mirroring

Some early reviewers didn't know how to "mirror" a line "about" a mirroring line. Here is one example.

Page 1 – The Problem



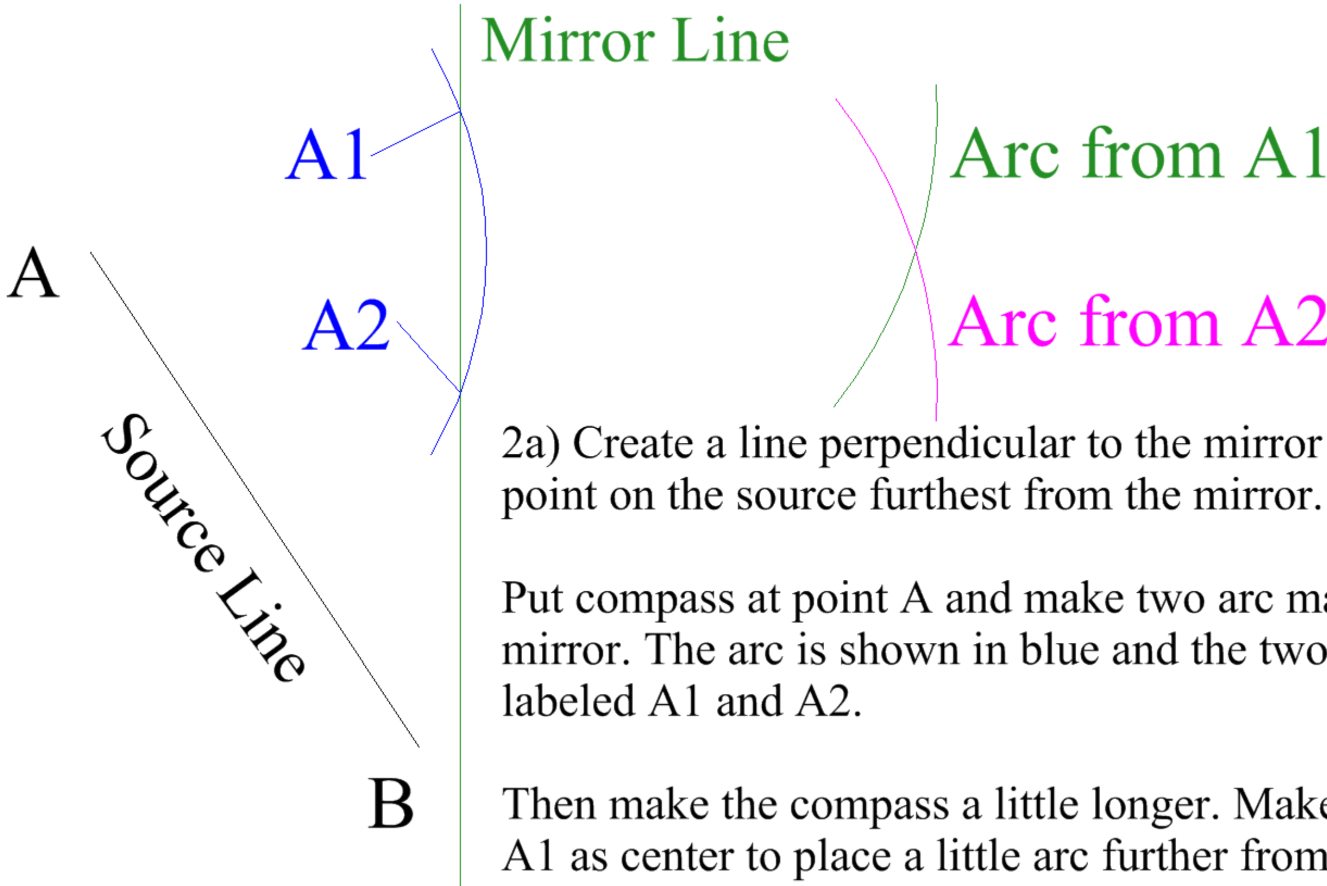
Given a line (shown in black) mirror it about another line (shown in green).

The "mirror" does not have to be vertical, but we'll do that for clarity. The mirror line may, or may not, touch or cross the source line. Here it does neither.

1) Make sure the mirror line is longer than both ends of the source line. Here it is so.

Appendix D - page 1 Mirroring A Line About Another Line

Page 2 – Point A



2a) Create a line perpendicular to the mirror that reaches the point on the source furthest from the mirror. How?

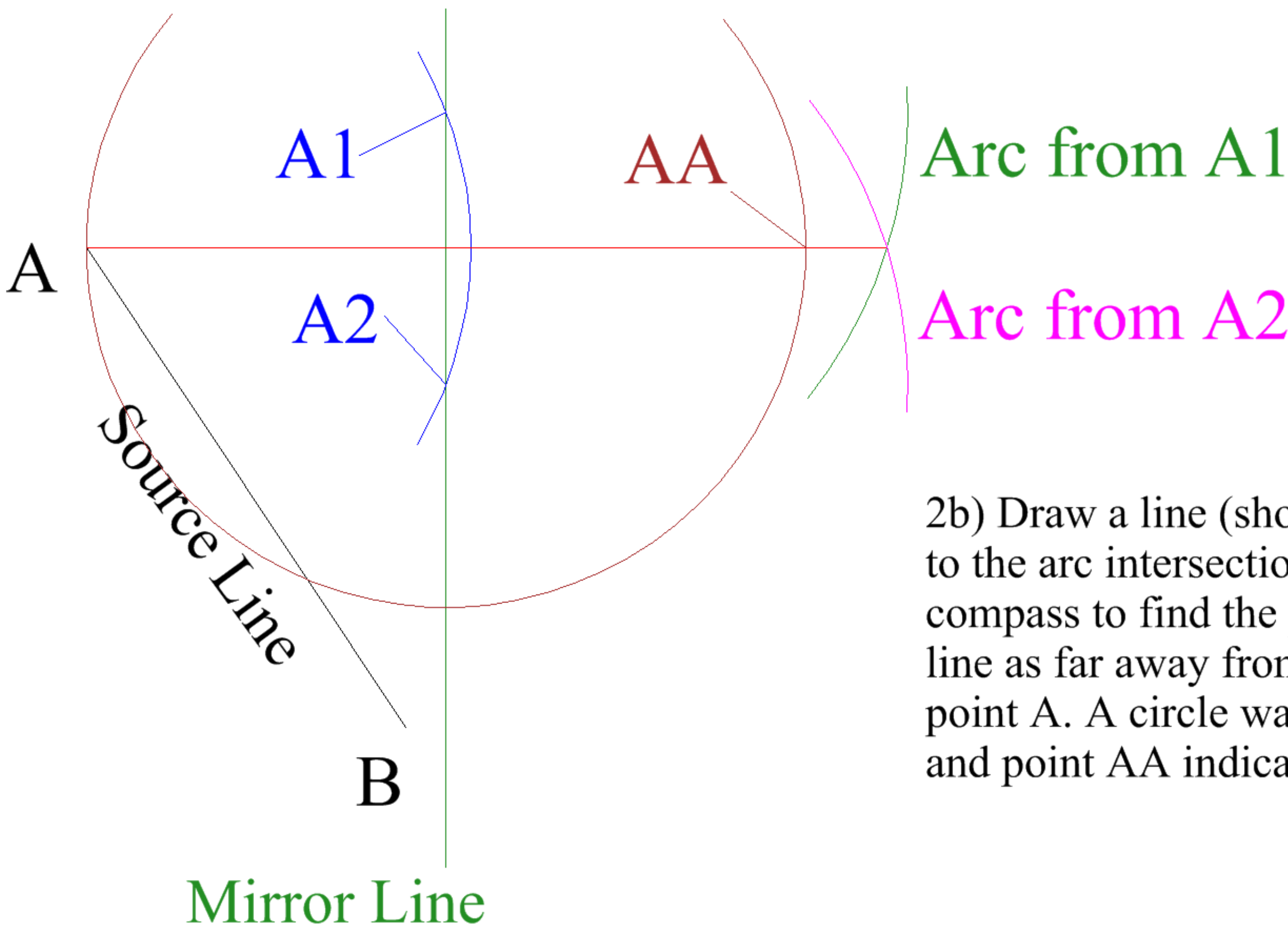
Put compass at point A and make two arc marks on the mirror. The arc is shown in blue and the two points are labeled A1 and A2.

Then make the compass a little longer. Make an arc with A1 as center to place a little arc further from the source line. Arc shown in green.

Using the same radius make an arc with A2 as center. Arc shown in magenta.

Appendix D - page 2 Mirroring A Line About Another Line

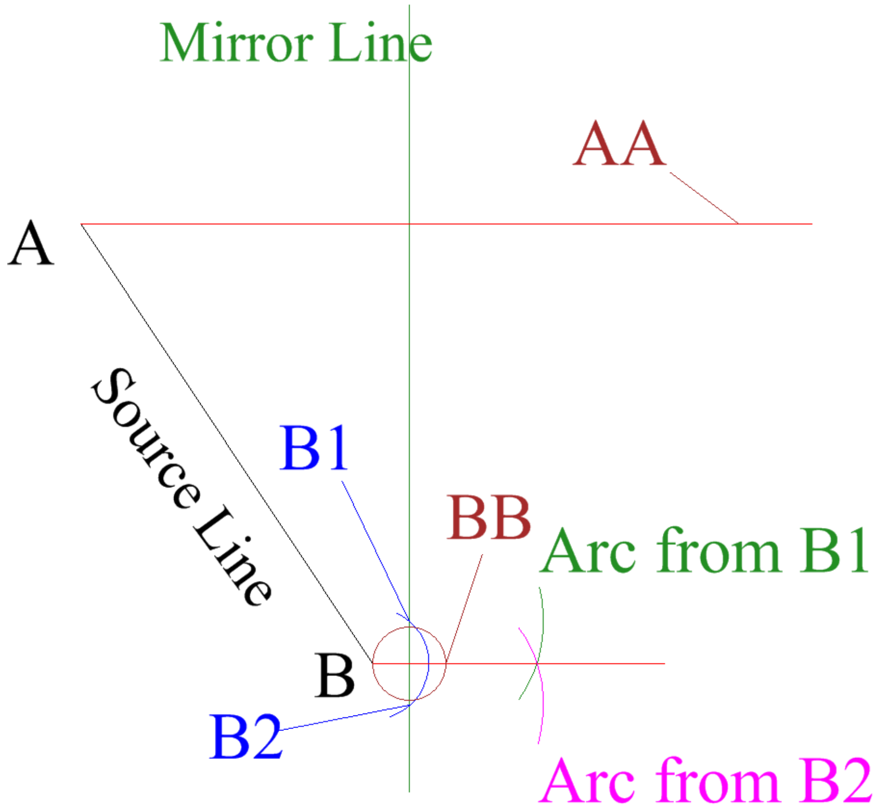
Page 3 – Point AA



2b) Draw a line (shown in red) from A to the arc intersections. Then use the compass to find the point on the red line as far away from the mirror as point A. A circle was drawn in brown and point AA indicated.

Appendix D - page 3 Mirroring A Line About Another Line - Point AA

Page 4 – Point BB

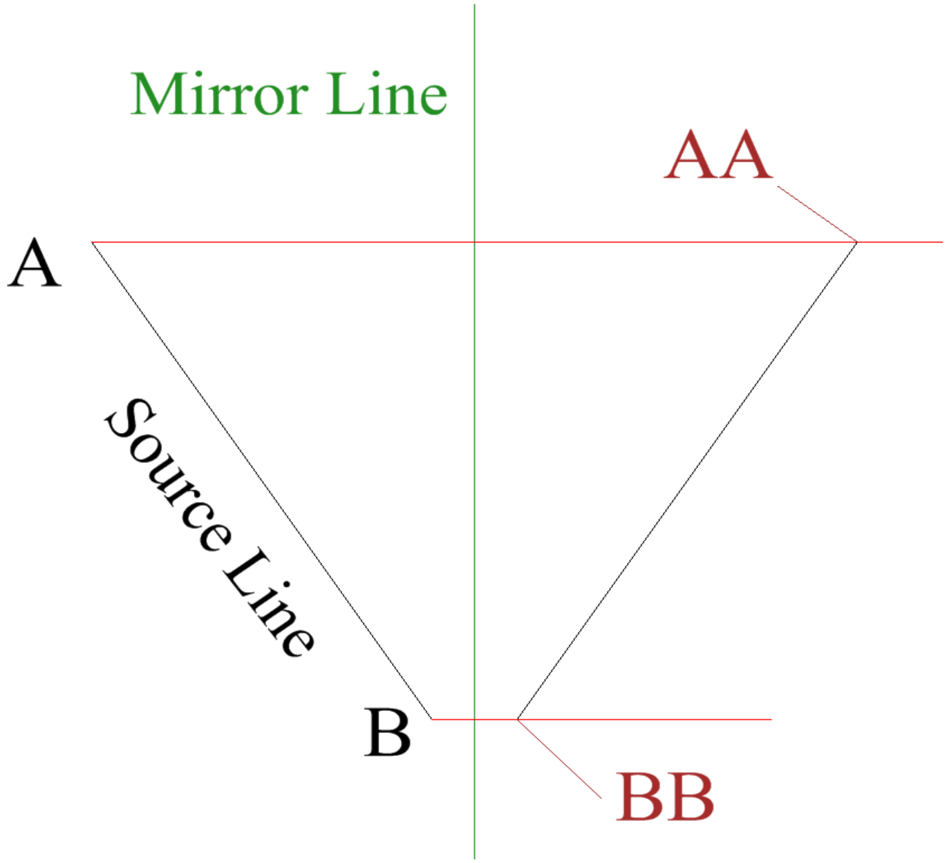


3) Do the same at point B to find point BB.

Appendix D - page 4 Mirroring A Line About Another Line - Point BB

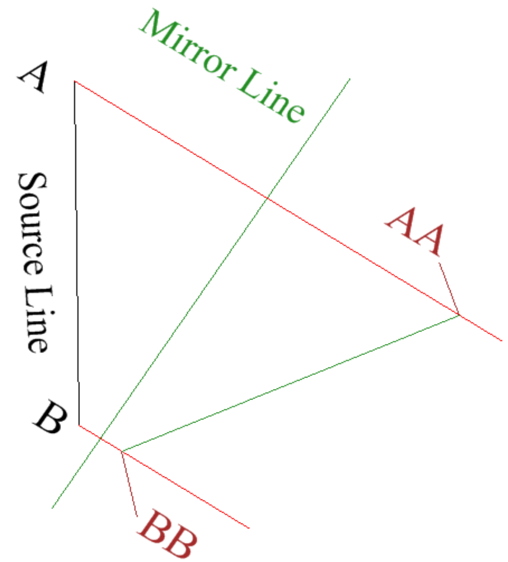
B1 and B2 are points on the mirror from an arc whose center is point B on the source.
The brown circle is centered on the intersection of the mirror and the red line with a radius to point B on the source.

Page 5 – Solution



4) Draw line AA-BB
This is the mirror of the line A-B mirrored about the green line.

Notice this construction is independent of orientation.
The same problem and solution is shown at a different angle.



Appendix D - page 5 Mirroring A Line About Another Line - Connect AA to BB

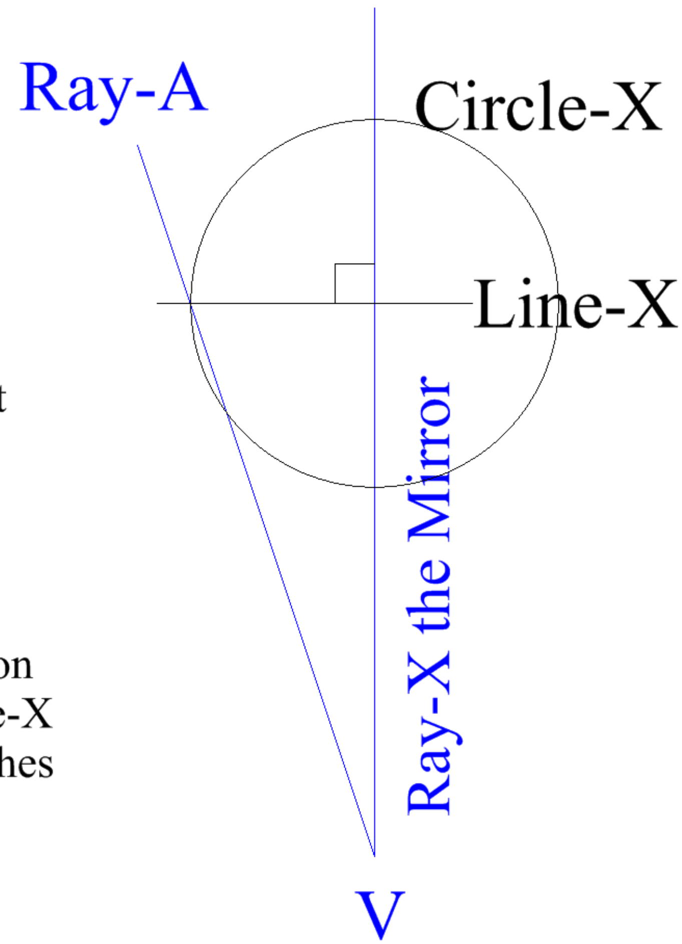
Page 6 – Mirroring Ray-A

Mirroring a ray is easier as each line (ray, angle bisector, or other) that eminates from the vertex has that point in common.

The problem is shown in blue.

Step 1 - create a perpendicular line at some point along Ray-X that extends to Ray-A. That is Line-X and it should extend a little past Ray-A. Extending Line-X to the right (toward where Ray-AA will be) happens later.

Step 2 - Draw Circle-X center at the intersection of Ray-X and Line-X and radius to where Line-X crosses Ray-A. You only need an arc that reaches where Ray-AA will be.



Appendix D - page 6 Mirroring Ray-A

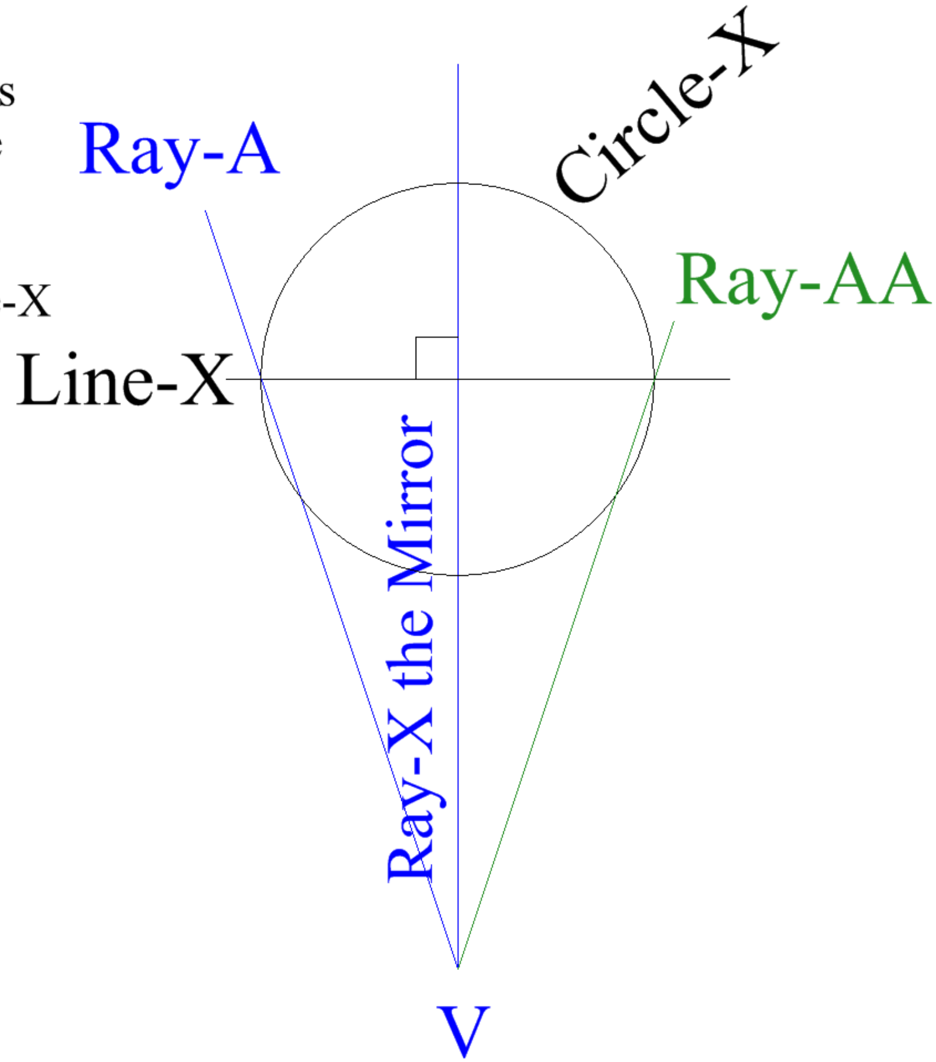
Page 7 – Ray-AA

Step 3 - Extend Line-X to the right to cross the perimeter of Circle-X on the other side of Ray-X.

Step 4 - Draw a line from V to where Line-X crosses the perimeter of Circle-X to the right side of Ray-X, the mirror.

That line is Ray-AA (shown in green) created by mirroring Ray-A about Ray-X.

As all eminations from the vertex extend to infinity make Ray-AA as long as you want.



Appendix D - page 7 Ray-AA

Appendix E

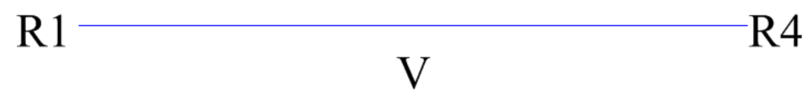
180° Special Case

Page 1 – As Before

A flat line ...

A flat line

... can be trisected by indicating the rays R1 and R4 then arbitrarily picking a vertex.

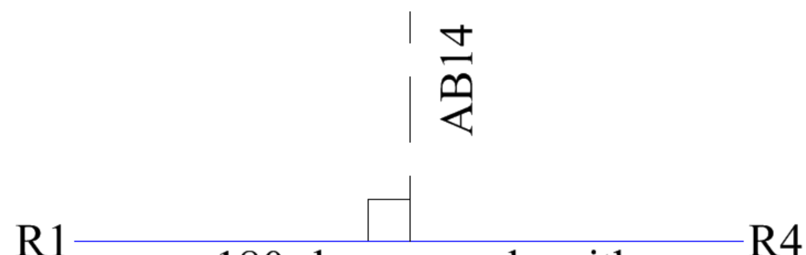


a 180-degree angle with identfied
Vertex and rays R1 and R4

Erect a perpendicular from V as an angle bisector named AB14.

Once you have V, R1, and AB14 you can proceed as described in the mainline presentation.

Or, remember something and use the shortcut described on the next page.



a 180-degree angle with
identified rays R1 and R4
and angle bisector AB14

Appendix E - page 1 Special Case

Page 2 – Shortcut

Cosine is Adjacent (line leading to right angle) divided by Hypotenuse (the diagonal line opposite from the right angle).

The Cosine of 60 degrees is 0.5 (1/2).

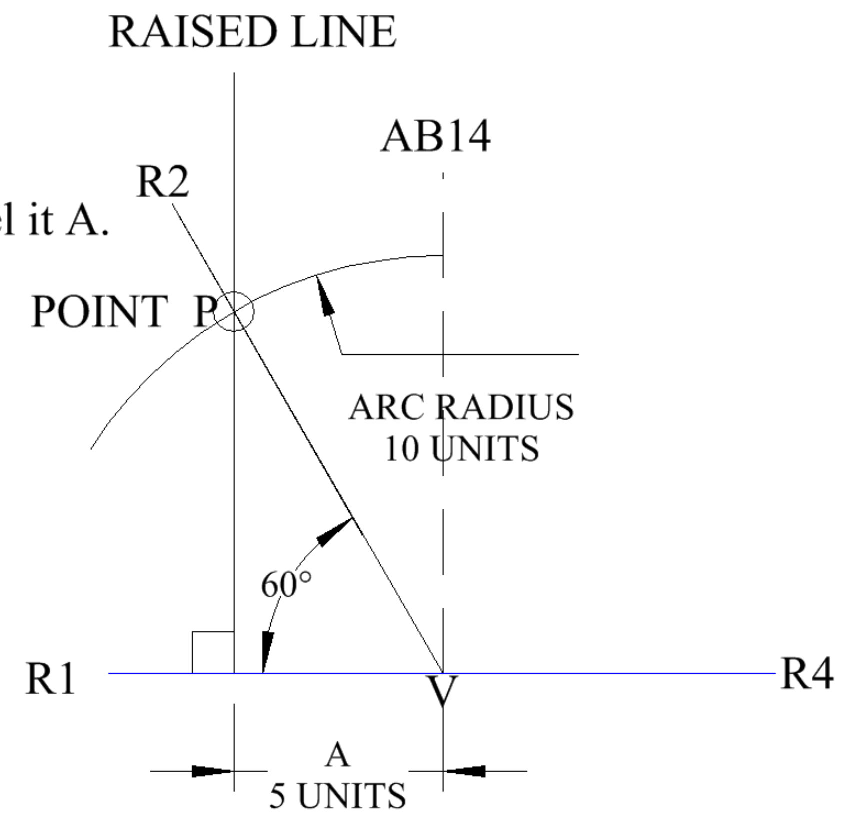
Draw a line from V toward R1, 5 units long, label it A.

Raise a line perpendicular from that point more than 10 units.

Draw an arc, center at V, radius 10 units and have that arc cross the raised line.

Draw a line from vertex to intersection of arc and that raised line.

Label the intersection P as shown.



That angle R1-V-P is 60 degrees 1/3 of the 180-degree angle, and R2.
Mirror R2 about AB14 to get R3.

Appendix E - page 2 - Special Case shortcut

Appendix F

Angles Greater than 180 Degrees

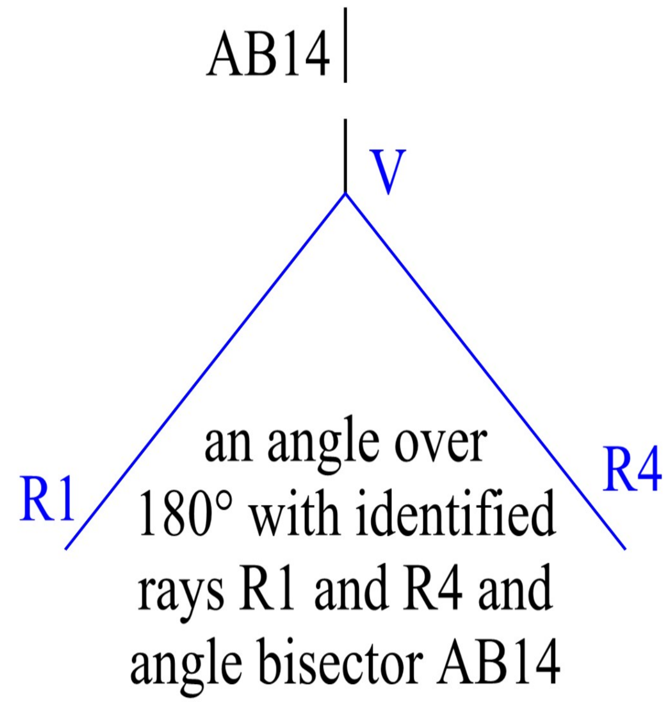
Page 1 – AB14

Does the process described work for angles greater than 180 degrees? Sure. Start by bisecting R1-V-R4 and create the angle bisector AB14 as before.

There is a third pair of tools that may be used.

Consider these possible reasons:

> a) Radius of Arc-a is less than three inches. Remember the arc is drawn thru the widest point of the last pad of the little finger. If you slide your little finger close to the vertex that might be too small for an effective radius.
>
> b) The chord is too long. You can use this pair of tools to reduce the gap between the two ants after two steps without having to move the little finger toward the vertex which reduces the radius of Arc-a.
>
> c) If the chord is too short (the little finger pad won't fit) after two ant steps don't use the new pair of tools, they will shorten the chord more. You may extend the rays from V as described in Appendix G starting on page 68.



Appendix F - Angles Over 180 Degrees
page 1 - AB14

Page 2 – Tools

Looking again at half the problem, create the tools as described in Appendix C.

Because this is a "wide" angle a third set of tools is recommended. Those are:
R3.125%, a ray at half the angle of R1-V-R6.25%
R1.5625%, a ray at half the angle of R1-V-R3.125%

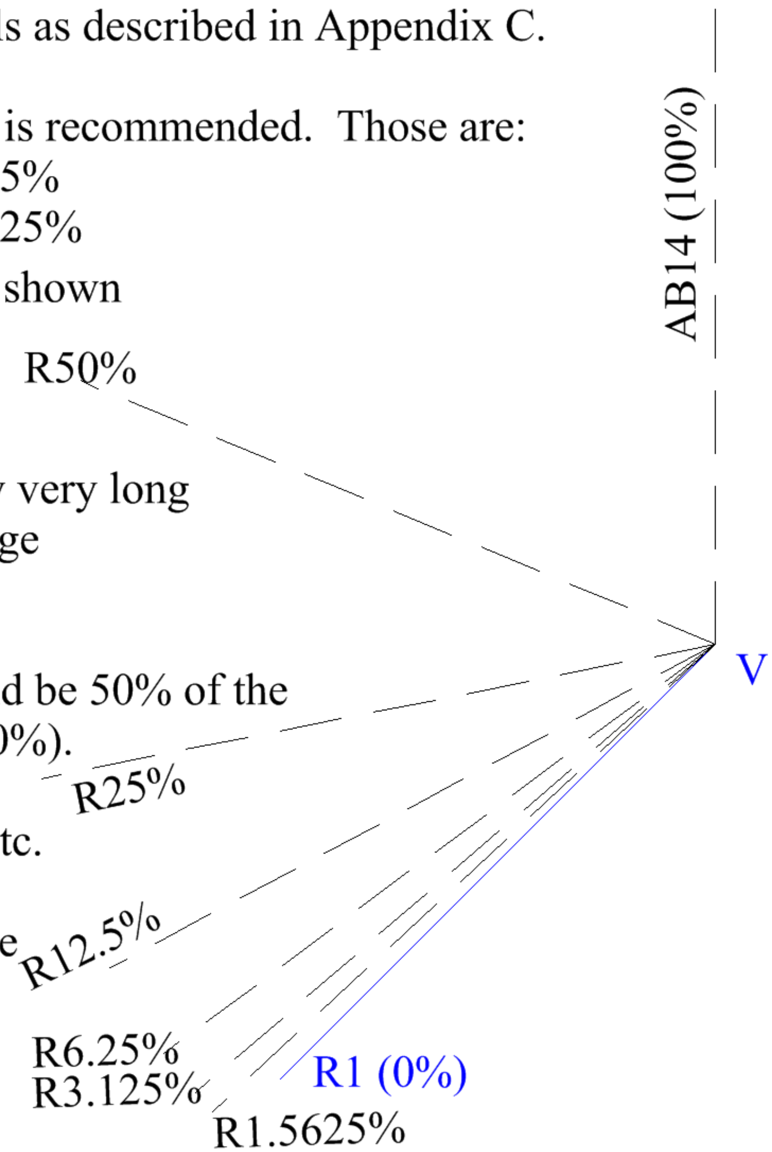
At construction these are all angle bisesctors and shown with dashed lines. You could make them all solid lines as other rays but really, why bother?

The appropriate label of the form ABnn gets very very long so just label them for their meaning as a percentage from R1 (zero percent) to AB14 (100%).

R50% is the bisector of R1 and AB14. That would be 50% of the way along an arc from R1(R0%) and AB14(R100%).

R25% is halfway between R50% and R1(R0%) etc.

Each angle is one-half of the next angle clockwise and twice the next angle counter-clockwise.



Appendix F - Angles Over 180 Degrees
- page 2 Create Tools

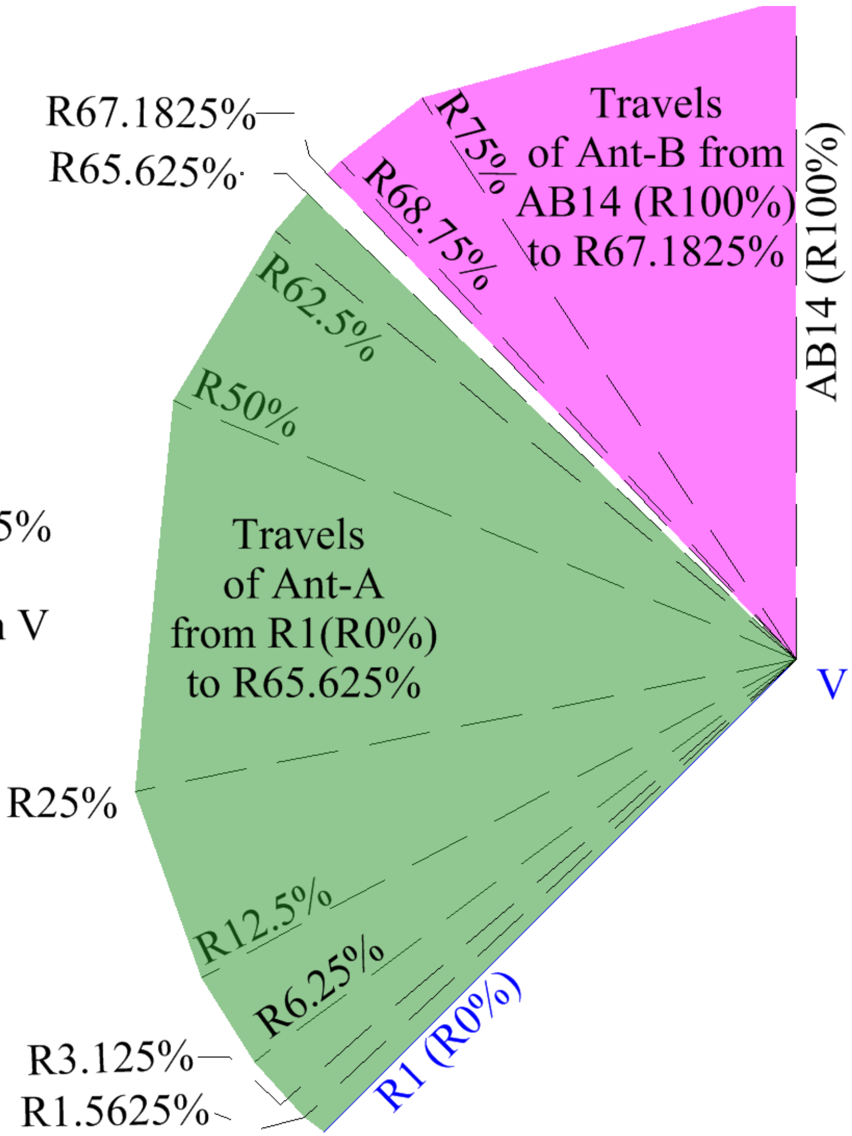
Create the three steps for each ant.

Ant-A starts from R1 (0%)
to R50% (step 1),
then R62.5% (step 2),
then R65.625% (step 3).

Ant-B starts from AB14 (100%)
to R75% then R68.75% then R67.1875%

The travel zones extend to infinity from V as do all rays and angle bisectors.

The travel zones are shown from V to a distance to fit here.



Appendix F - Angles Over 180 Degrees
- page 3 Ant Steps

Page 4 – Arc-A Chord Trisection

Create Arc-a, center at V. Now about the radius . . . the longer the radius the more workspace you get between Ant-A and Ant-B travels. At some point you need really large sheets of paper. I used radius as shown to create Arc-a.

Create a chord
from the intersection of Arc-a and R65.625%
to the intersection of Arc-a and R67.1825%.

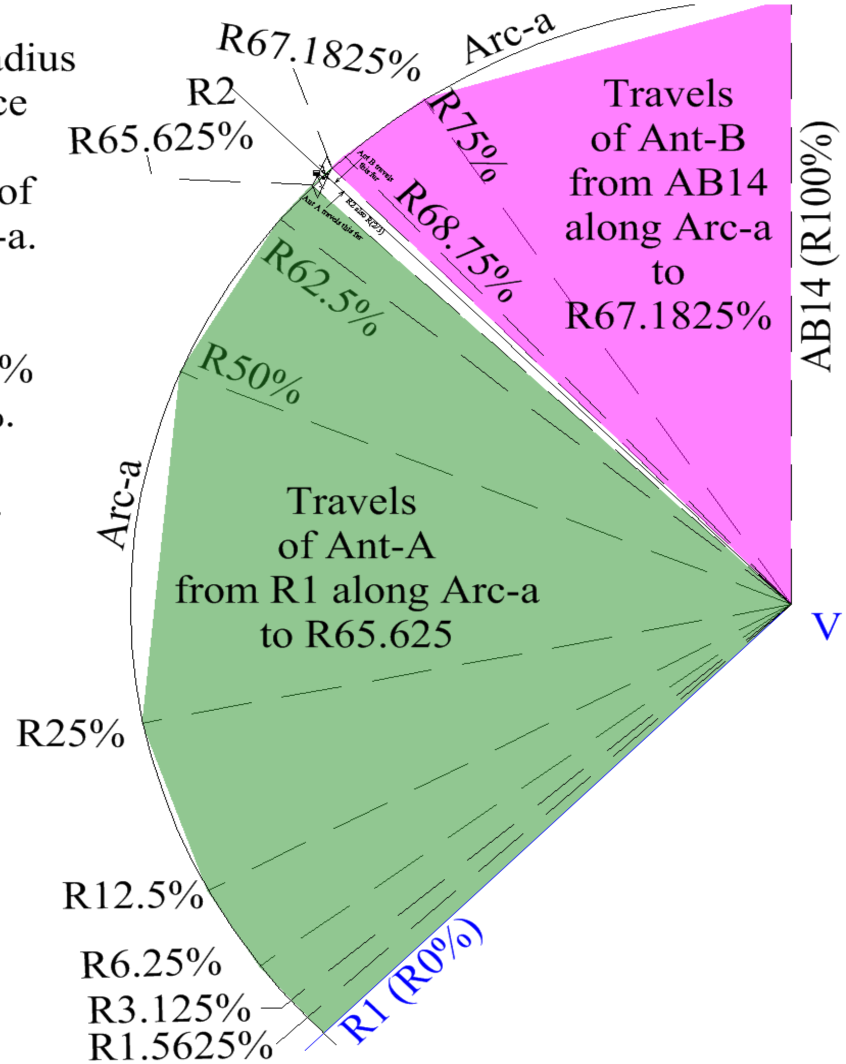
Trisect that chord as shown in Appendix A.

Draw a line from V to where the counterclockwise most 2/3 meets the clockwise most 1/3.

That line is R2.

An enlarged image of the trisection area is on page following.

Appendix F - Angles Over 180 Degrees - page 4 Arc-a, Chord, Trisection



Not enough room to do the chord trisection?

See Appendix G on page 68 for an optional technique to get more paper space.

Page 5 – Trisection (magnified)

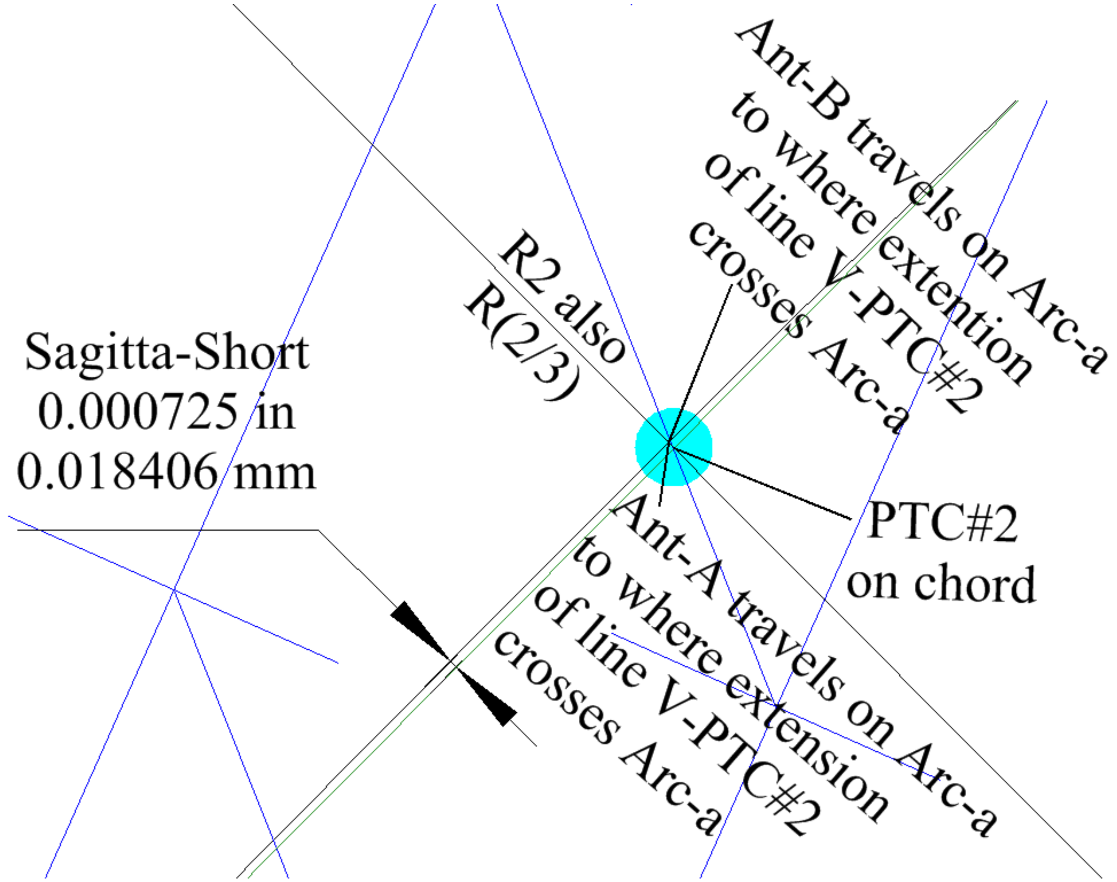
This is a closeup of the chord trisection area. The blue lines are part of the chord trisection construction.

It represents the last step of both ants to where they meet on Arc-a. At each step Ant-A travels twice as far as Ant-B. Their meeting is at R2, the goal.

R2 would also be R66.666etc% that being the percentage of the angle R0%-V-R100%. The fractional form would be R(2/3)% or simply R(2/3).

The Arc-a radius was measured at 3.412015 inches. The chord length was 0.140625 inches. The Sagitta-Short was measured at 0.000725 inches or 0.018406 mm.

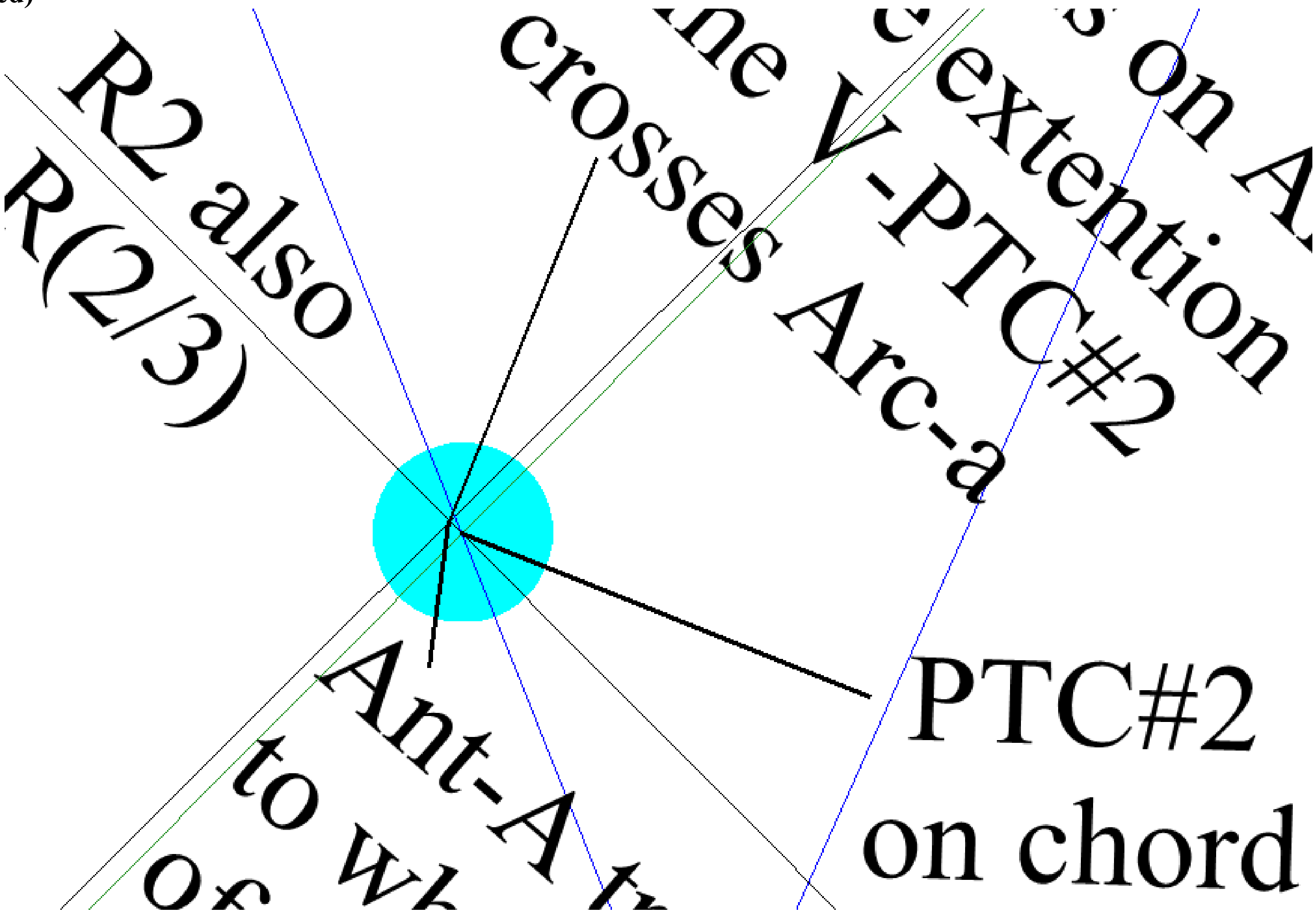
See next page for even greater magnification.



Appendix F – Angles Over 180 Degrees - Page 5 Trisection Magnification

If this was the whole problem, instead of half, then R2 would be R(1/3) being a third of the arc distance from R1 (R0%) to R4 (R100%). See both Appendix Y and Appendix Z.

Page 6 – Hairline (magnified)



The distance perpendicular from the center of the chord to the arc is 7.25 ten-thousandths of an inch. That is the maximum distance between arc and chord and about the diameter of a thick human hair[41].

The sagitta-short is 0.018406 mm, less than 20% of a 0.1 mm line width. The sagitta-short would be obscured by a line centered on the chord and another line centered on the arc.

Any line thickness over 0.018406 mm would render the difference between the arc and chord infinitesimal.

Appendix F – Angles Over 180 Degrees – Page 6 Hairline Magnification

The blue lines are part of the chord trisection construction.

Page 7 – 0.1mm line (magnified)

infinitesimal /ĭn″fĭn-ĭ-tĕs′ə-məl/

an adjective meaning:

1) Immeasurably or incalculably minute.

2) Capable of having values approaching zero as a limit.

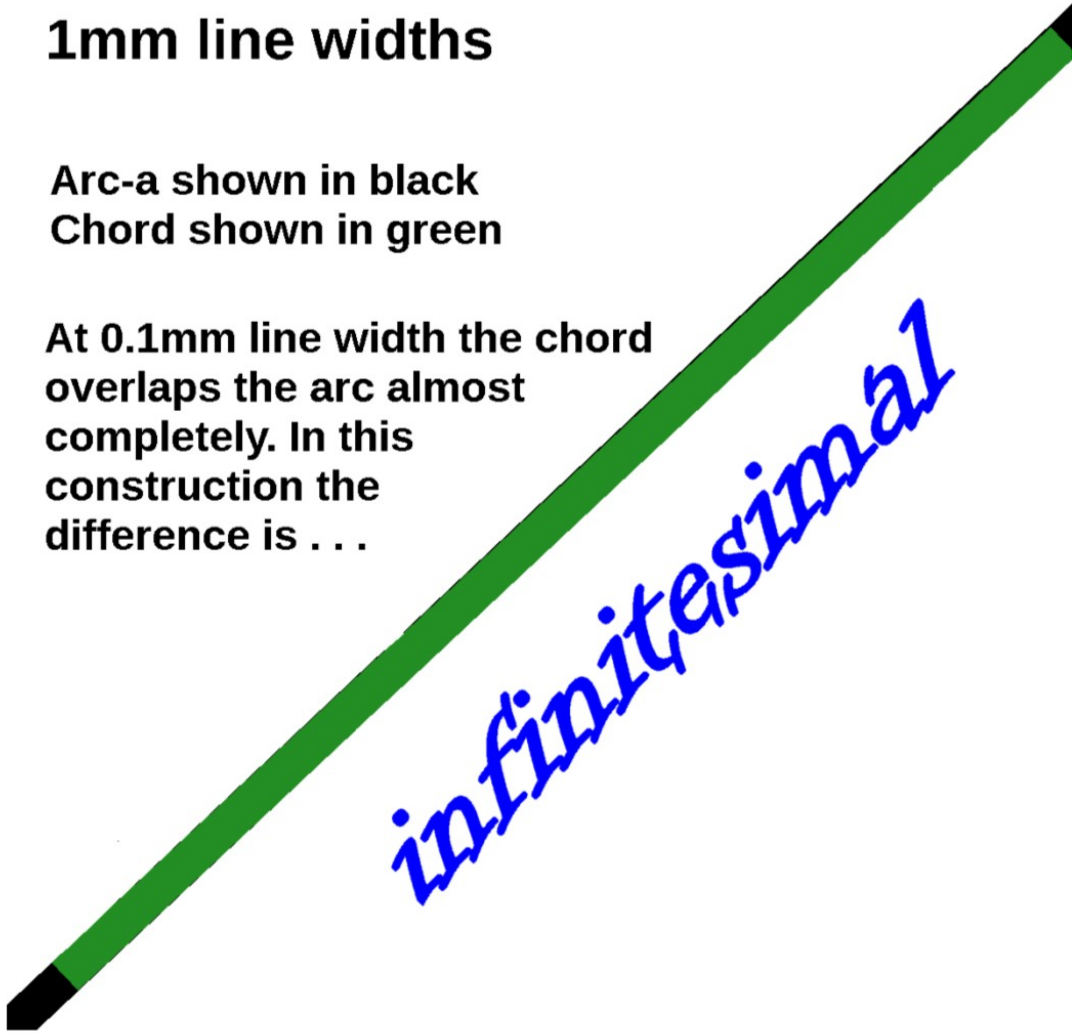
3) Infinitely or indefinitely small; less than any assignable quantity or value; very small.

The American Heritage® Dictionary of the English Language 5th Edition

1mm line widths

Arc-a shown in black
Chord shown in green

At 0.1mm line width the chord overlaps the arc almost completely. In this construction the difference is . . .



Appendix F - Angles Over 180 Degrees
- page 7 Magnification (0.1 mm line thickness)

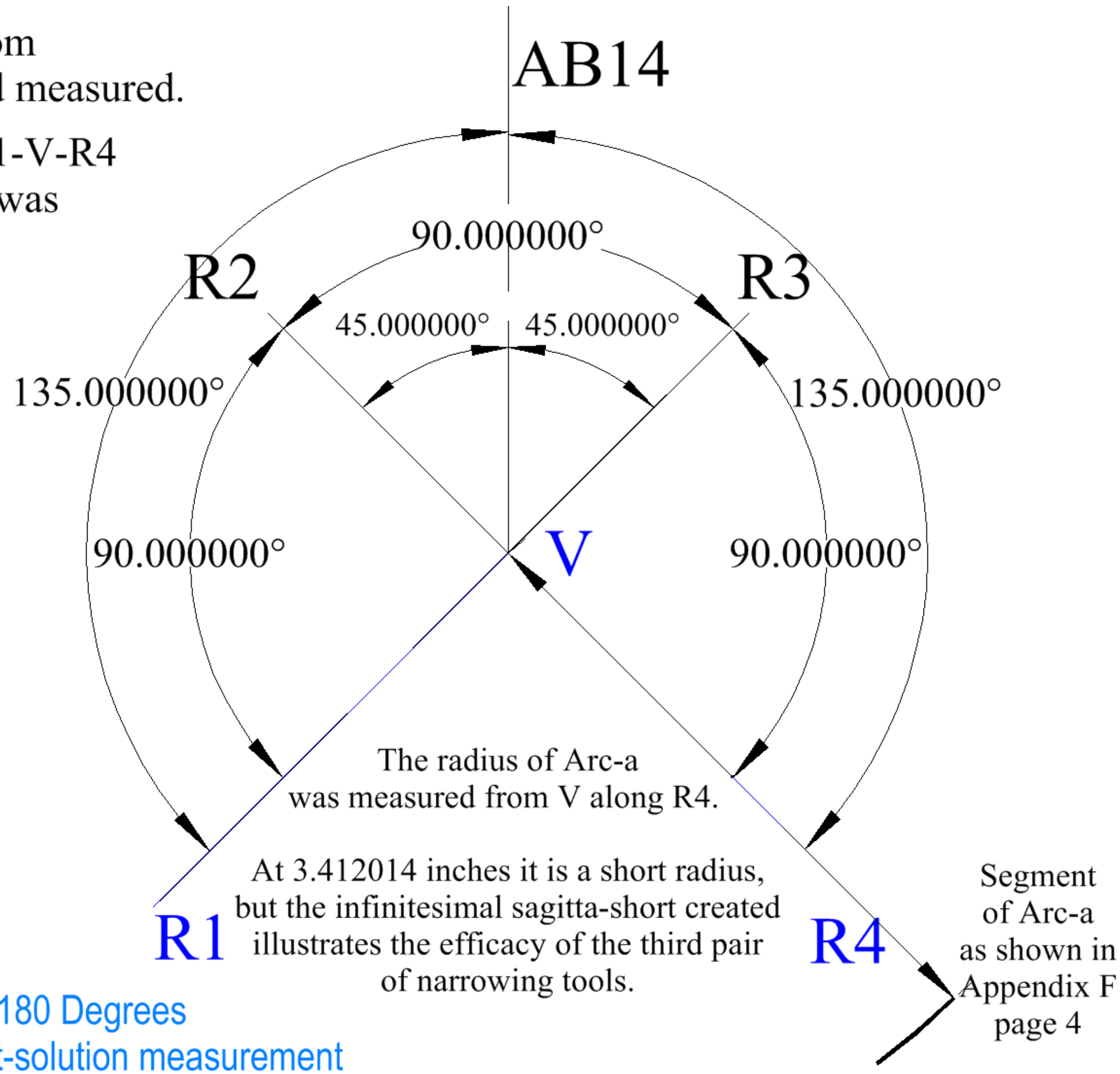
I copied the solution from Appendix F, Page 4 and measured.

Given only the angle R1-V-R4 we found R2. Then R2 was mirrored about AB14 to find R3.

Afterward measured to find R1-V-AB14 was 135-degrees

so R1-V-R4 was 270-degrees

a third of 270 is 90 and that is where R2 was located.



Appendix F - Angles Over 180 Degrees
- page 8 Solution and post-solution measurement

Appendix G

Thick Fingers

An optional technique to trisect the chord at a place with more finger room

Page 1 – Extending Rays

For two triangles “similar” means that the three interior angles in one triangle are the same interior angles as are in the other triangle.

An early reviewer had problems doing the line trisection on the chord because the chord was small (and on purpose).

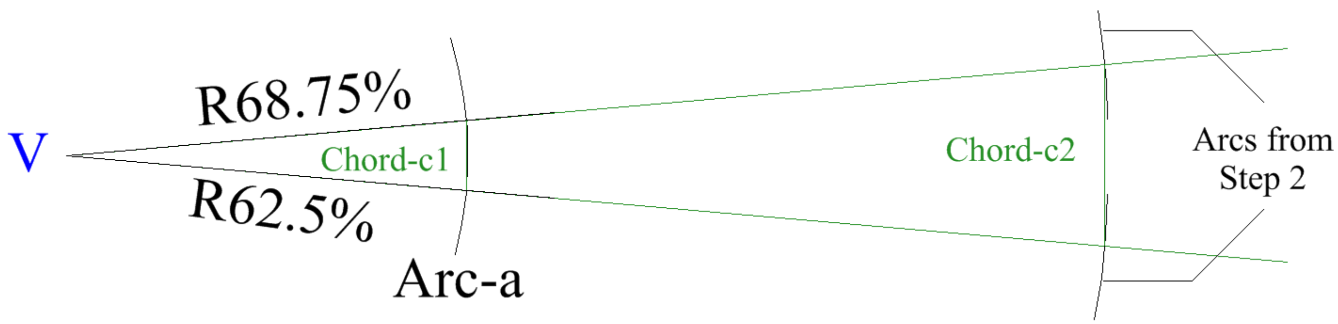
Reminder: That which radiates from the V for Vertex goes to infinity.

To get some more room to work on trisecting the chord start with a familiar part of a solution just prior to trisecting the chord:

1) Extend the two rays so there is more room to do the line trisection and maybe a little bit more. Extensions shown in green.

2) Using the compass mark a distance from Arc-a along one ray then the same distance on the other ray. Marks shown in black.

3a) Draw a line from the intersection of each ray and the distance mark to the intersection of the other ray and its distance mark. Label this Chord-c2. Notice there is no arc similar to Arc-a.



Re 3a: Being “similar” triangles the Sagitta-Short to the Arc-Radius ratio is the same at Chord-c1 and Chord-c2.

Appendix G - page 1 - Not enough room
Extend Chord-c1 to Chord-c2

3b) Chord-c2 is Chord-c1 copied away from the vertex. The chord is enlarged by the increasing ray separation as Chord-c2 moves further from the vertex.

Similarly the Sagitta-Short at Chord-c2 would be longer than the Sagitta-Short at Chord-c1. Shorter is better so Chord-c2 is used only for trisection assistance of Chord-c1 This is why there is no arc at Chord-c2 similar to Arc-a at Chord-c1.

Page 2 – Trisect c2

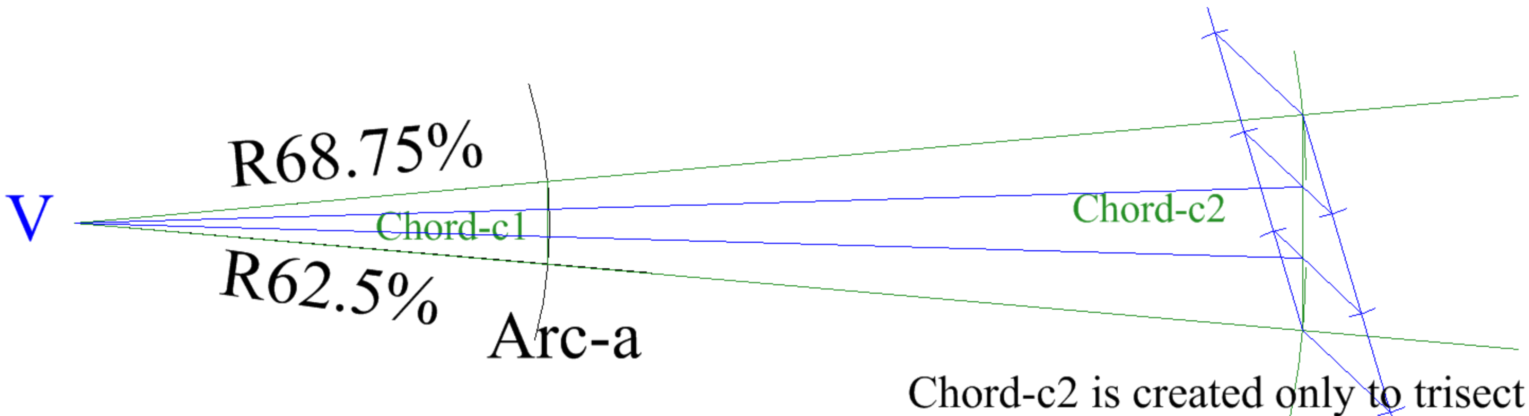
4) Trisect Chord-c2 as described in Appendix A. Trisection is shown in blue.

5) Draw a line from V to each trisection point of Chord-c2. Those lines are also in blue.

6) As each of the blue lines passes through Chord-c1 they create trisections of Chord-c1. See magnifications on following pages.

The extension Chord-c2 creates a triangle similar to that created by Chord-c1.

For each triangle the two interior angles are equal and each is half of 180-degrees less the common angle R68.75%-V-R62.5%.



Appendix G - page 2 - Not enough room
Trisect Chord-c2

Chord-c2 is created only to trisect Chord-c1. The sagitta-short of Arc-a and Chord-c1 is different than Chord-c2 and another arc spanning the two rays.

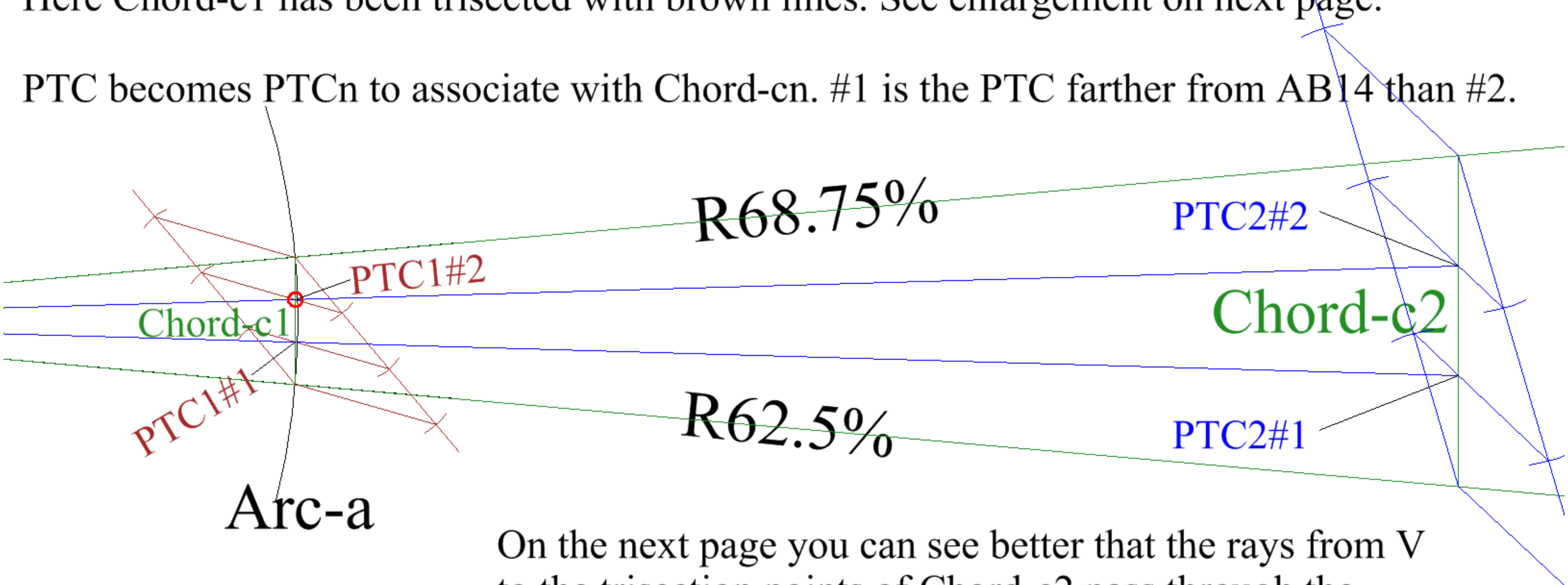
The “trisection points” on Chord-c2 are more fully described on the next page.

Page 3 – c1 & c2 (magnified)

For clarity the distance marks from Arc-a are not shown here.

Here Chord-c1 has been trisected with brown lines. See enlargement on next page.

PTC becomes PTCn to associate with Chord-cn. #1 is the PTC farther from AB14 than #2.



On the next page you can see better that the rays from V to the trisection points of Chord-c2 pass through the same PTC1#1 & PTC1#2 on the trisected Chord-c1.

Appendix G - page 3 - Not enough room
Trisected Chord-c1 and Chord-c2 (magnified)

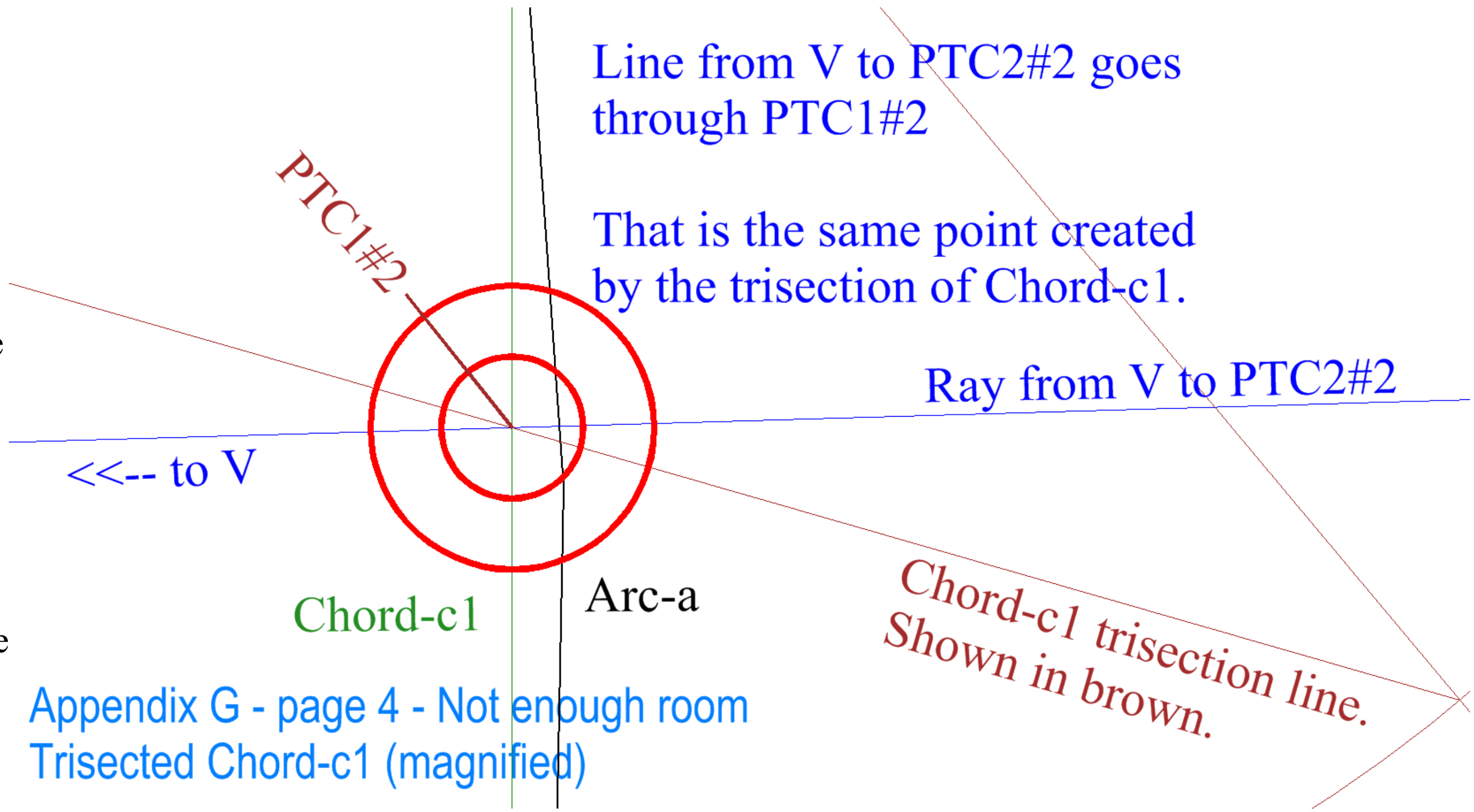
See next page for a further magnification showing Chord-c1 and its trisection done with the brown lines.

Page 4 – c1 (magnified)

Where the extended line V-PTC2#2 crosses the arc (shown in black) is where Ant-A meets Ant-B and that is R2.

PTC points are on the chord. The extended line from V to the PCT crosses the arc where ants travel.

Why is PTC1#2 and PTC2#2 higher on the page than PTC1#1 and PTC2#1?



Look back at Appendix G page 1. R68.75% is above R62.5% indicating that R100% is above the view presented. Recall that PTC#2 is closer to R100% than PTC#1.

(Some small notes buried near the end: I'd try to start with an R1 and R4 length equal to a spread hand, from thumb tip to little finger tip, about 5 to 7 inches. Using a little-finger-last-pad of about half-an-inch makes for a good sized sagitta-short. Short rays? Remember that which emanates from the vertex runs to infinity so extend the rays and change the conditions of the test. Really long ray? Use less of it with the same spread hand. If arithmetic, inspiration, and geometry will do the job why complicate it?)

Appendix Y

Why Oh Why?

Didn't I notice this decades ago??

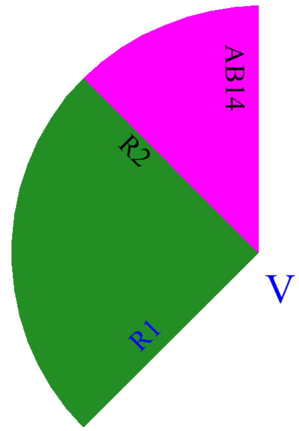
Page 1 – Initial

I started by changing the conditions of the test by working with only half the problem and got an inspiration. The green area was twice as large as the magenta area. The green was one equal third (ET) and the magenta was one-half ET.

I uniquely identify the ET angles clockwise as ET1, ET2, and ET3. The angle is the same for ET1, ET2, and ET3. i.e. ET1 = ET2 = ET3.

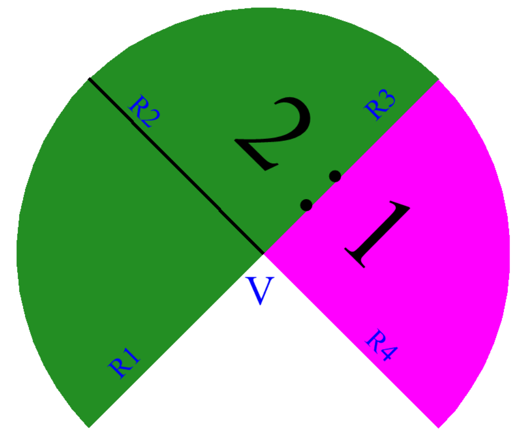
So the initial inspiration can be noted as

ET1(green) : ET2/2(magenta)
multiply both sides by two to get
2xET1(green) : 1xET2(magenta)
Divide both sides by ET
to get **2(green) : 1(magenta)**



Recall the initial inspiration was dividing the angle in half to see one section twice as large as another.

Appendix Y - Page 1 - Initial Inspiration



Why didn't I see the whole problem this way?

I suspect that it was partly neuro-linguistic programming (NLP) whereby repetitive words become ingrained in our brain and influence the way we think. This was always "three equal thirds".

So, can the concept still work for the whole angle?

Getting mentally away from “three equal thirds” it is evident that the “whole problem” situation can be noted as

(ET1(green) + ET2(green)) : ET3(magenta)
all three ETn being equal substitute with plain ET, aggregate the greens, and get
2xET(green) : 1xET(magenta)
2(green) : 1(magenta) the same ratio.

The procedure works on the whole problem as demonstrated below.
Why didn’t I see that earlier? Why did no one else see that earlier? Why oh why?

Page 2 – Whole Problem

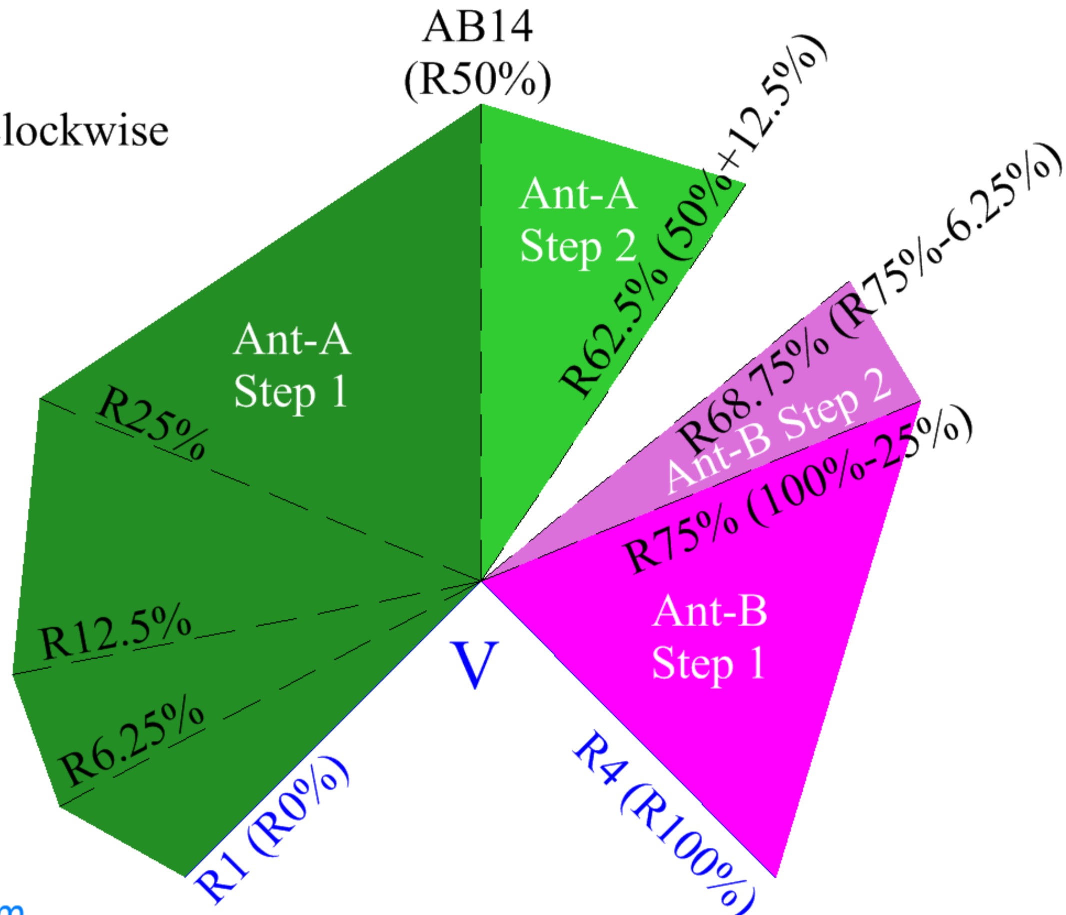
Some changes:
R4 becomes R100% (AB14 was R100%)
AB14 becomes R50% (AB14 was R100%)
Ant-B starts at R4 (Ant-B was starting at AB14)

The goal is to find R3 (was R2)
to keep the clockwise/counterclockwise
motion of Ant-A and Ant-B

Create AB14, R25%, R12.5% and
R6.25% as before.

Show first two steps for each ant.

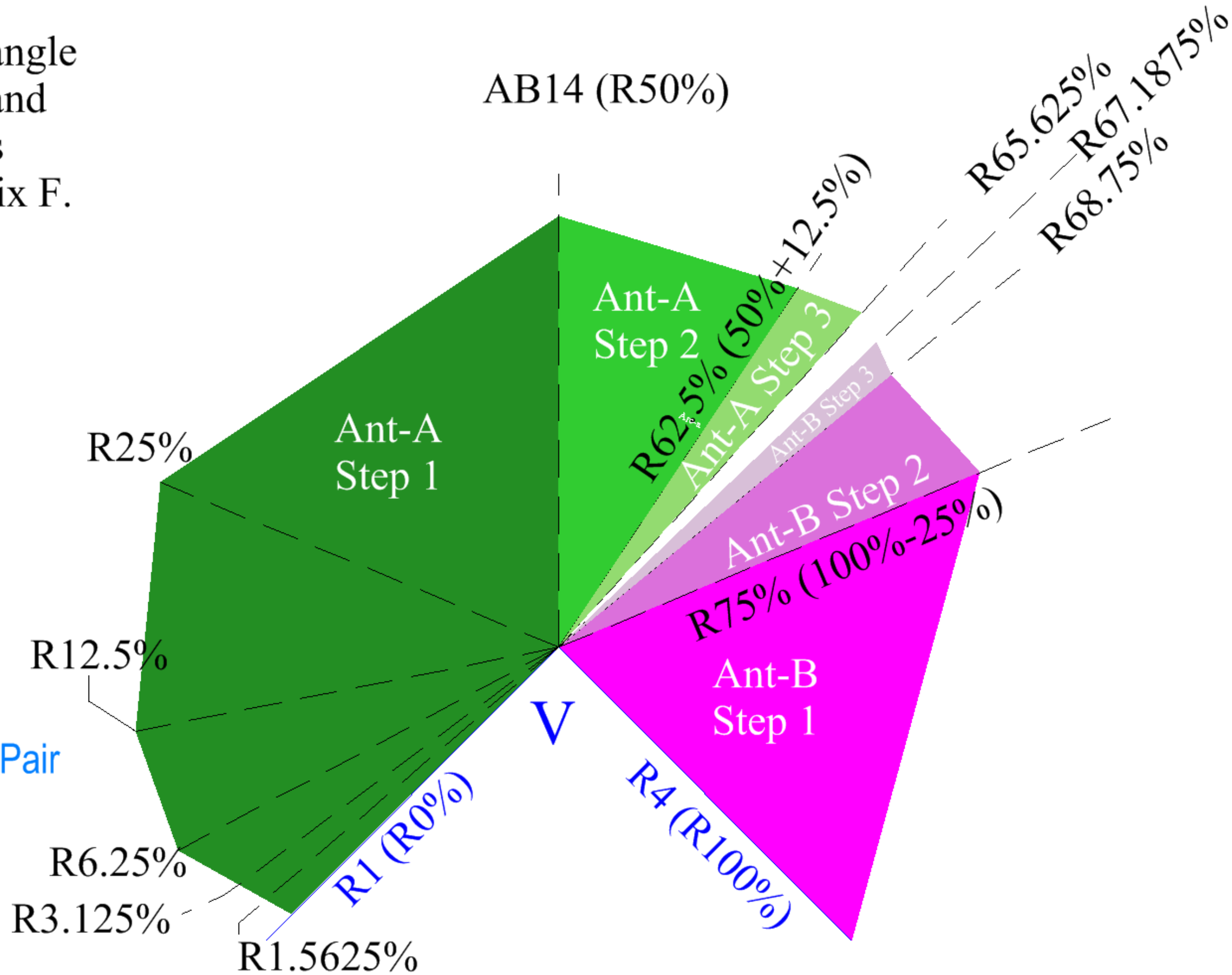
Remember: the travel zones,
like all rays and bisectors,
run from the vertex to infinity.
Those zones are shown at
a size to fit on the page.



Appendix Y - Page 2 - The Whole Problem

Page 3 – Third Pair

Because this is a large angle
we skip the finger pad and
use the third set of tools
as described in Appendix F.



Appendix Y - Page 3 - 3rd Pair

Page 4 – Arc-a & Chord

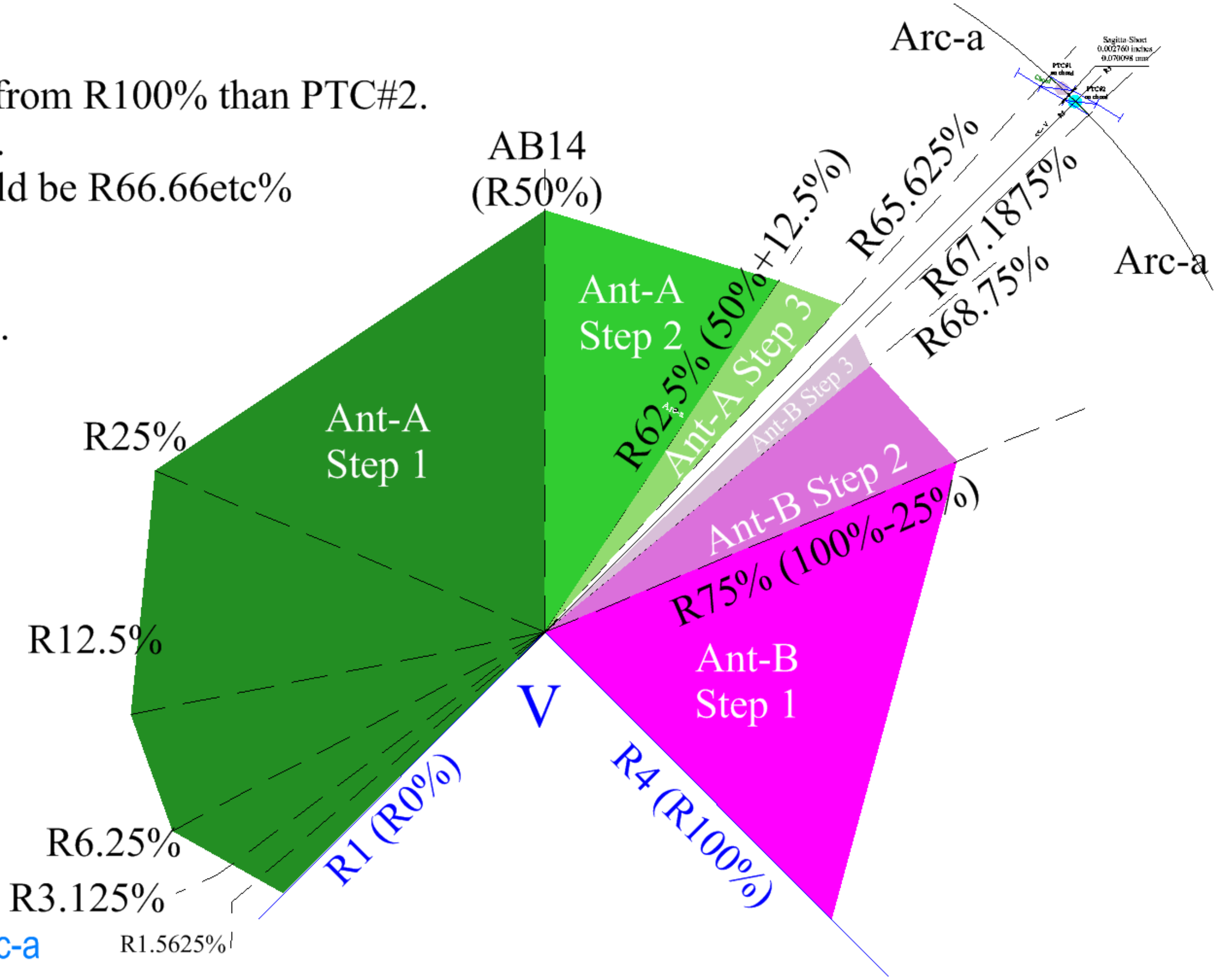
Remember:
PTC#1 is further away from R100% than PTC#2.
PTC#2 is a point on R3.
The line V-PTC#2 would be R66.66etc%

We put Arc-a beyond the colored travel zones.

Draw chord from intersection of Arc-a and R65.625% to intersection of Arc-a and R67.1875%

Trisect the chord
Draw R3

See enlargements on following pages.



Appendix Y - Page 4 - Arc-a

Page 5 – Magnification

PTC#1 is in the pinkish circle.

PTC#2 is in the blue circle.

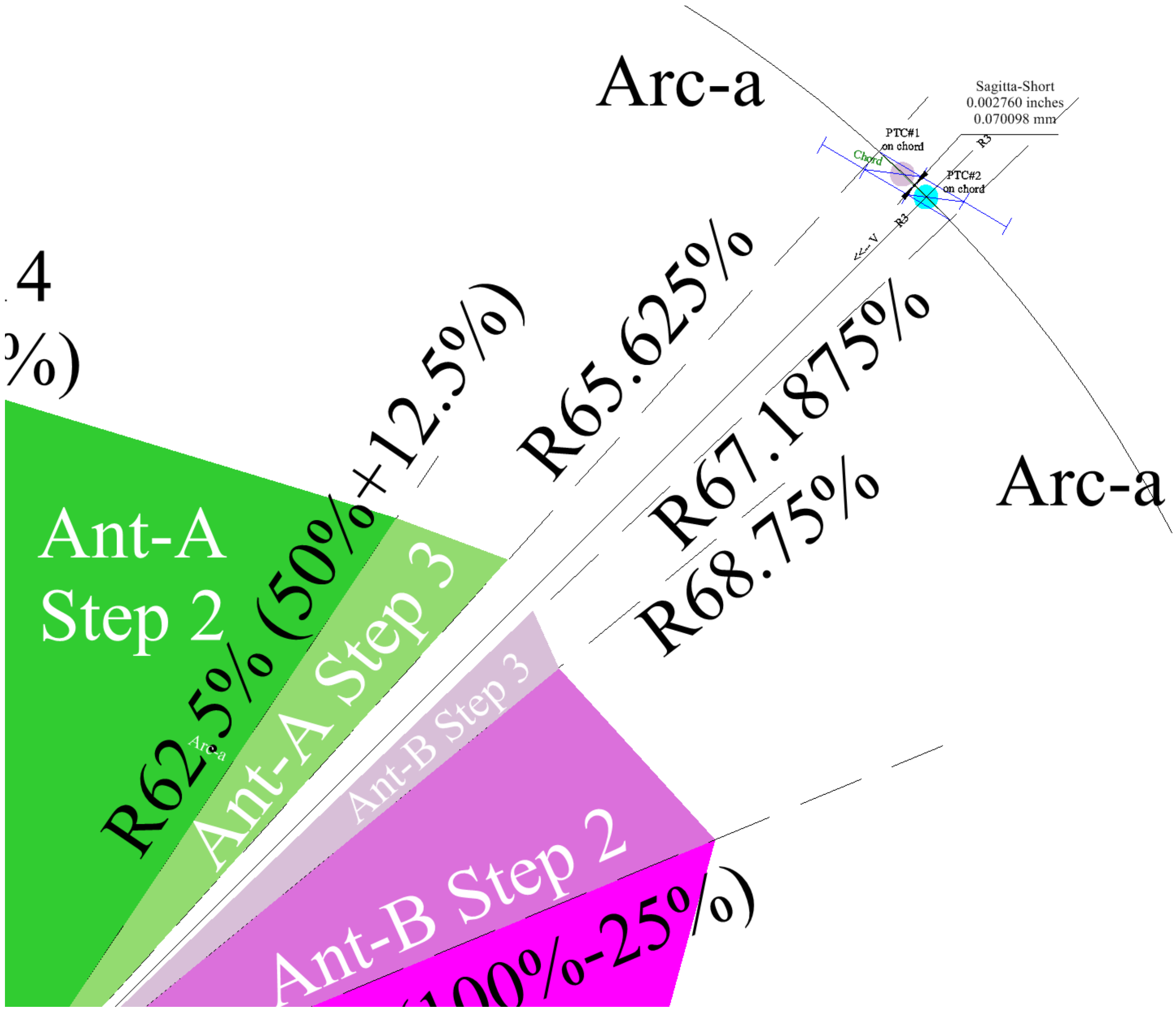
On Arc-a Ant-A moves from the light green Step 3 zone to a point where extended line V-PTC#1 crosses Arc-a (line not shown) and from there to where V-PTC#2 crosses Arc-a.

Ant-B moves from the light magenta Step 3 zone to where the line V-PTC#2 extends to cross Arc-a. The ants move along the arc.

The thirds of the arc are the same size to an infinitesimal difference.

So Ant-A moves twice the distance along the arc as Ant-B.

They meet on Arc-a at R3.



Page 6 – Trisection area

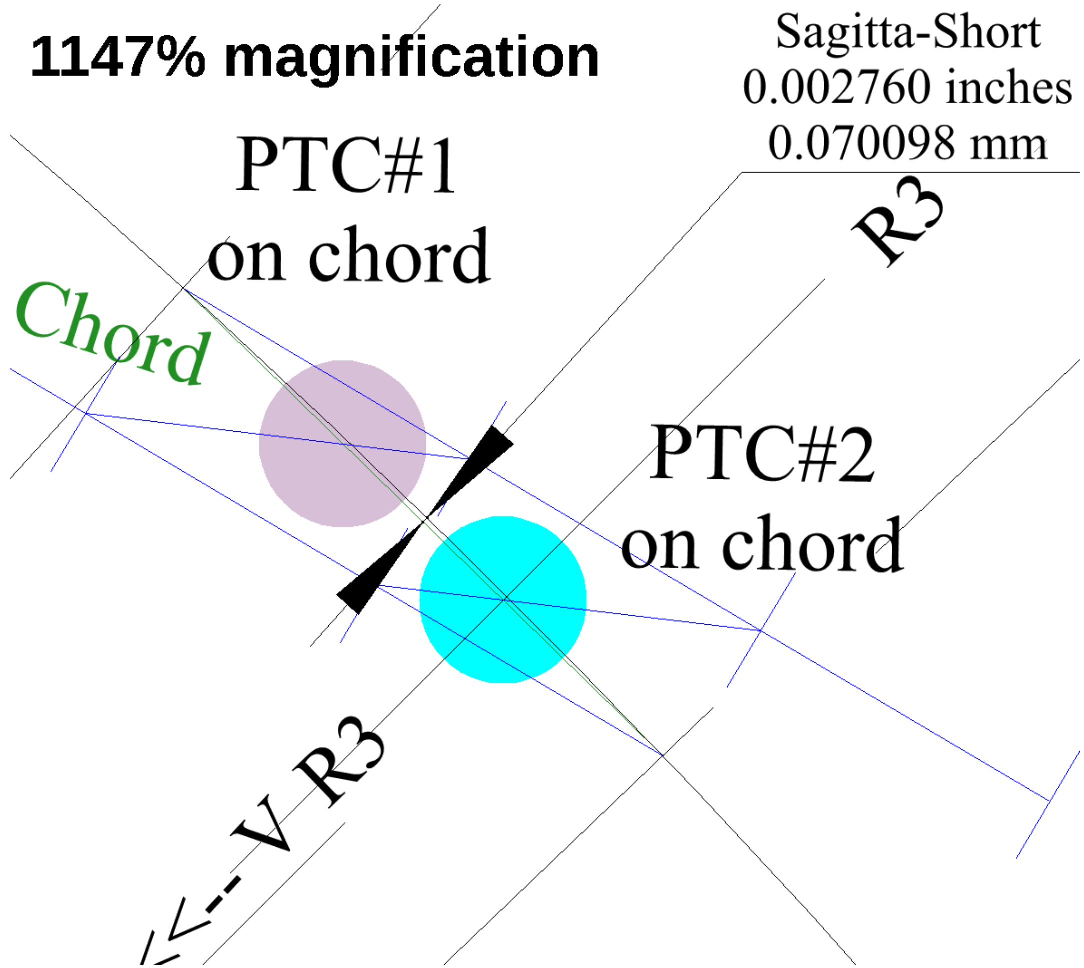
Arc-a was made at even less that the average spread of fingers with a radius measured at 4.068407”. The chord was measured at 0.299494”. The resulting sagitta-short was measured and calculated to six decimal places as 0.002760 inches or 0.070098 mm.

This means any line width of 0.070098 mm or wider drawn centered on the arc and centered on the chord will obscure the sagitta-short.

The difference between the chord and the arc ranges from 0.0 mm where both ends of the chord touch the arc to a maximum of 0.070098 mm which is between the 0.017 mm diameter of thin human hair and the 0.181mm diameter of thick human hair.

The third pair of tools was effective as was creating Arc-a at a distance to fit the paper.

The angle R1-V-R4 was 270.000°.
R4-V-R3 was 90.000° as expected.



The thin blue lines are chord trisection constructions as described in Appendix A.

The solution works on the whole problem, not just half.

Appendix Z

Simpler

Previous examples showed building the tools in the area R1-V-R50% and copying them as necessary. This was primarily for educational purposes to guide a mindset toward the solution and avoid a long-standing rut.

There is an simpler way ...
build them where they are needed.

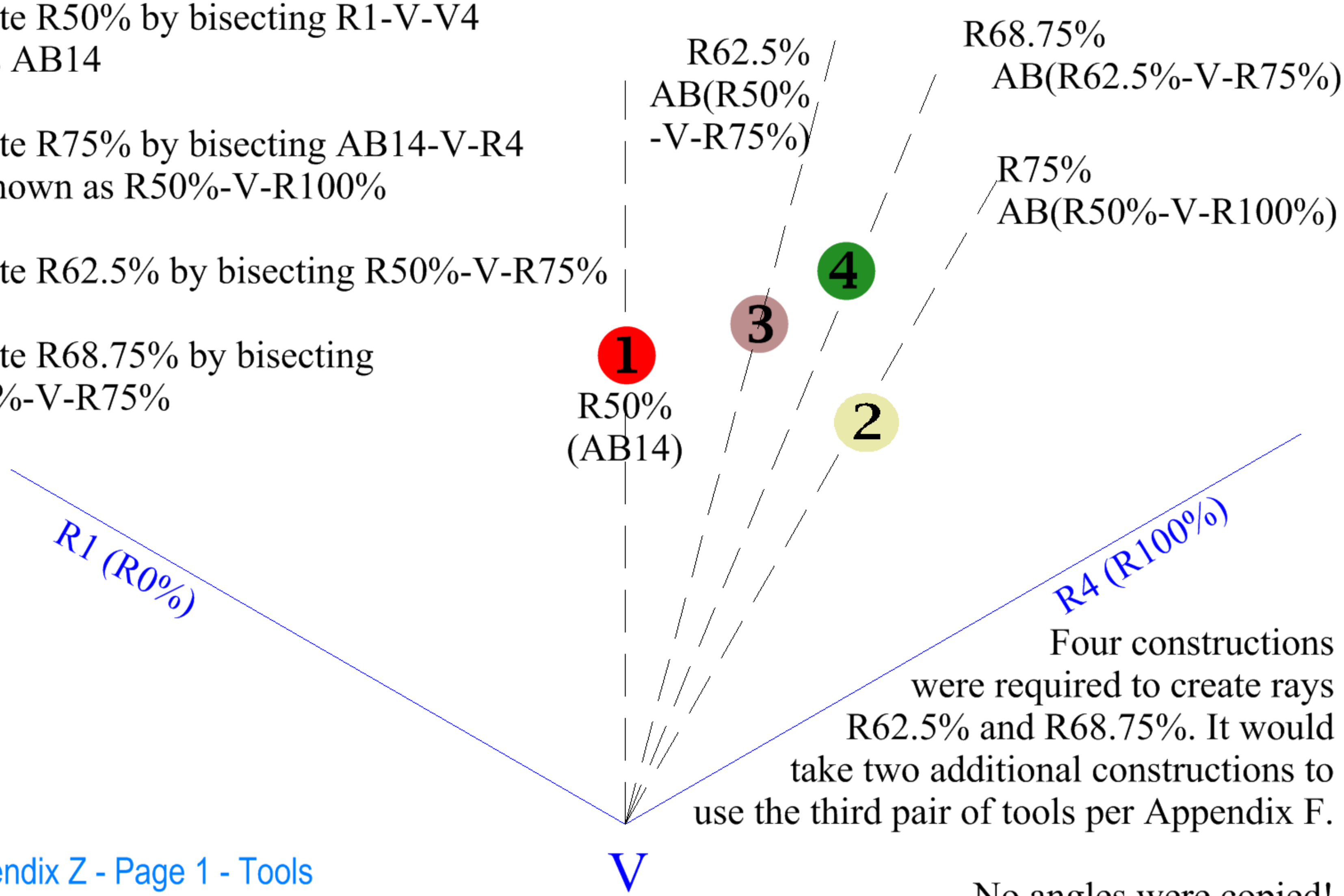
Page 1 – Tools

1 Create R50% by bisecting R1-V-V4
This is AB14

2 Create R75% by bisecting AB14-V-R4
also known as R50%-V-R100%

3 Create R62.5% by bisecting R50%-V-R75%

4 Create R68.75% by bisecting
R62.5%-V-R75%



Four constructions were required to create rays R62.5% and R68.75%. It would take two additional constructions to use the third pair of tools per Appendix F.

No angles were copied!

Appendix Z - Page 1 - Tools

Page 2 – Angles

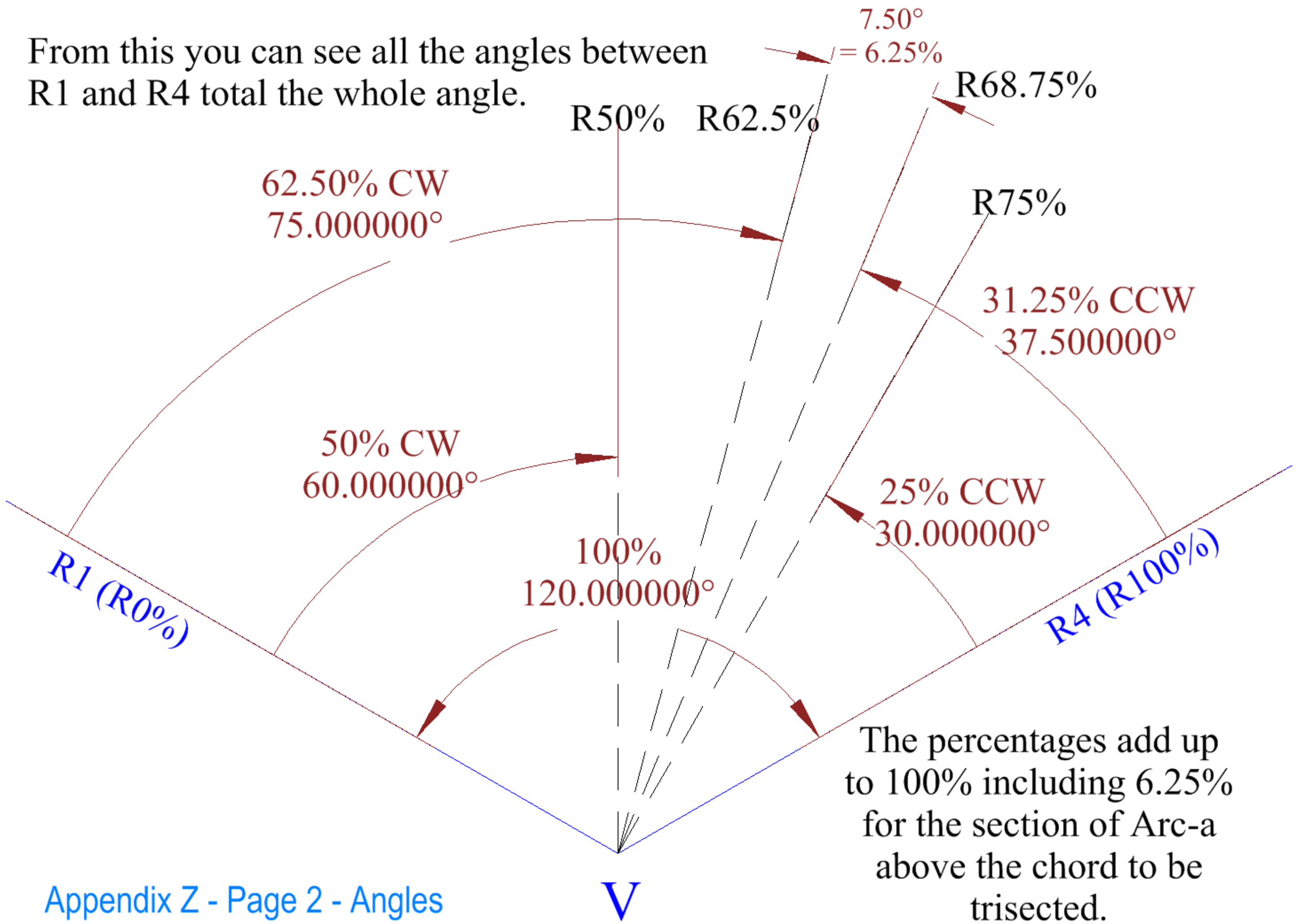
CW is clockwise.

CCW is counterclockwise.

After trisection the 6.25% will be 2/3 clockwise and 1/3 counter-clockwise so both Ant-1 (traveling CW) and Ant-2 (traveling CCW) meet at R3, one of the two rays unknown at the start of the problem.

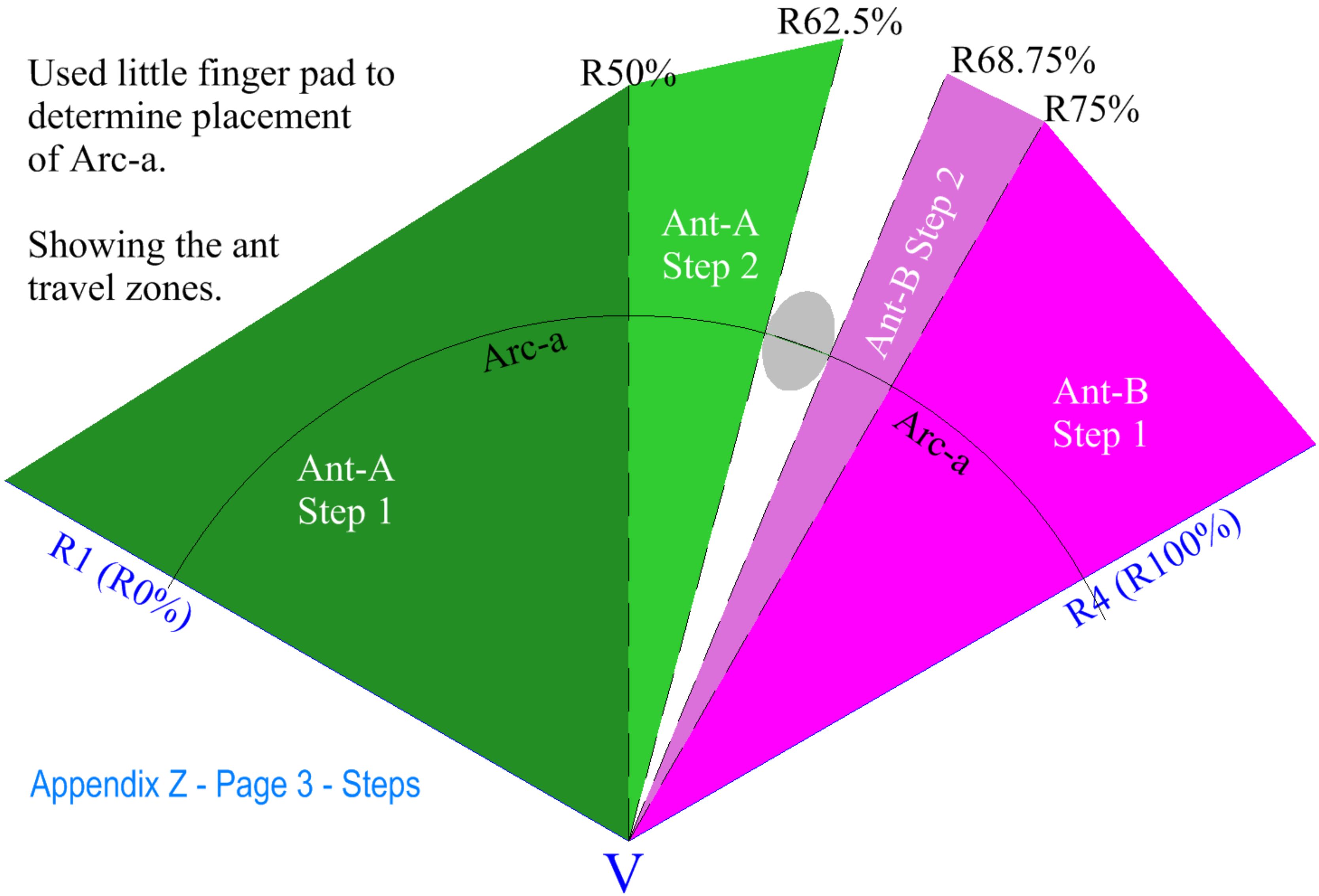
Mirror R3 about R50% (AB14) to find R2.

From this you can see all the angles between R1 and R4 total the whole angle.



The percentages add up to 100% including 6.25% for the section of Arc-a above the chord to be trisected.

Page 3 – Steps

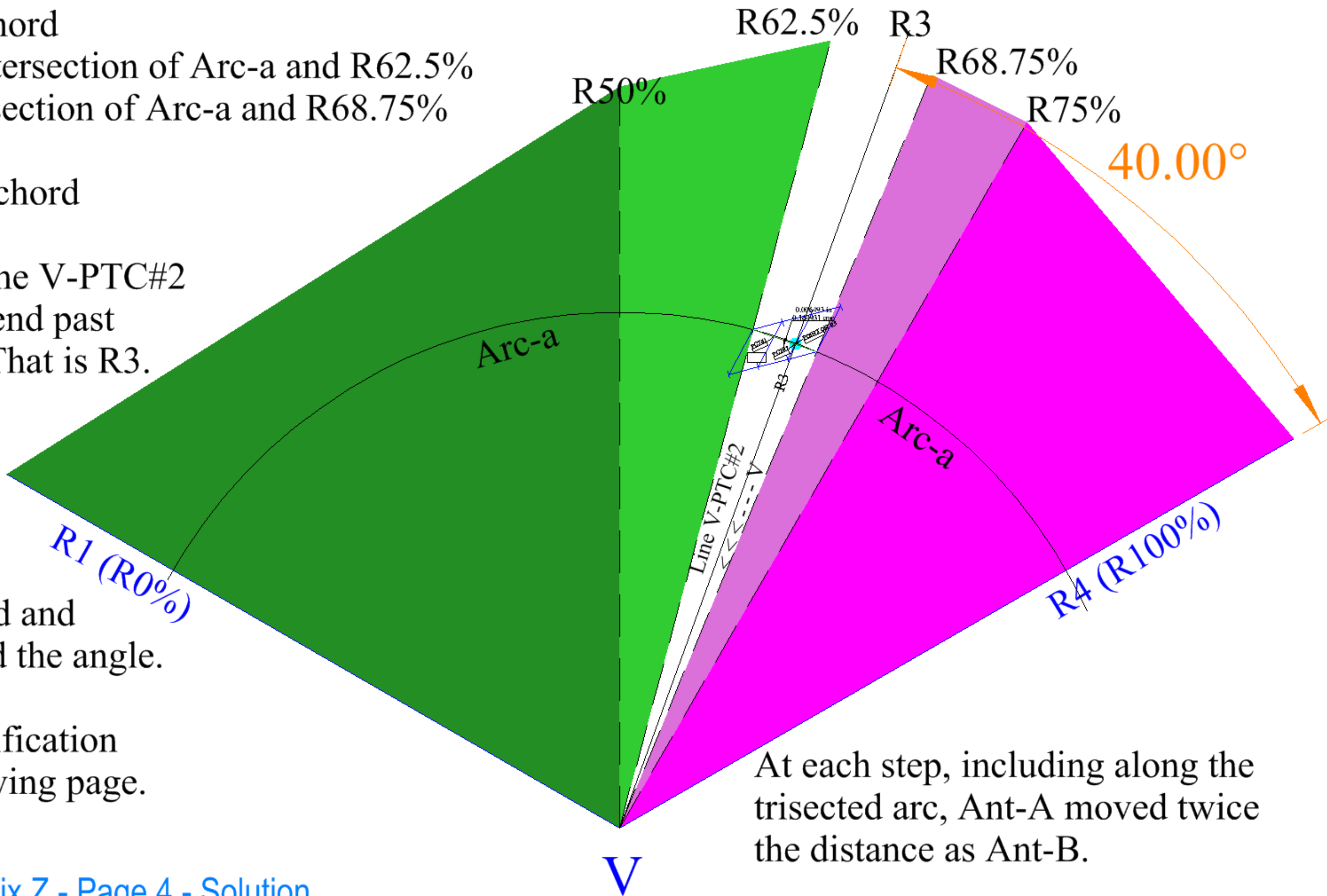

Used little finger pad to
determine placement
of Arc-a.
Showing the ant
travel zones.
R50%
R62.5%
R68.75%
R75%
Ant-A
Step 2
Ant-B Step 2
Arc-a
Arc-a
Ant-A
Step 1
Ant-B
Step 1
R1 (R0%)
R4 (R100%)
Appendix Z - Page 3 - Steps
V

Page 4 – Solution

Draw chord
from intersection of Arc-a and R62.5%
to intersection of Arc-a and R68.75%

Trisect chord

Draw line V-PTC#2
and extend past
Arc-a. That is R3.



Measured and
displayed the angle.

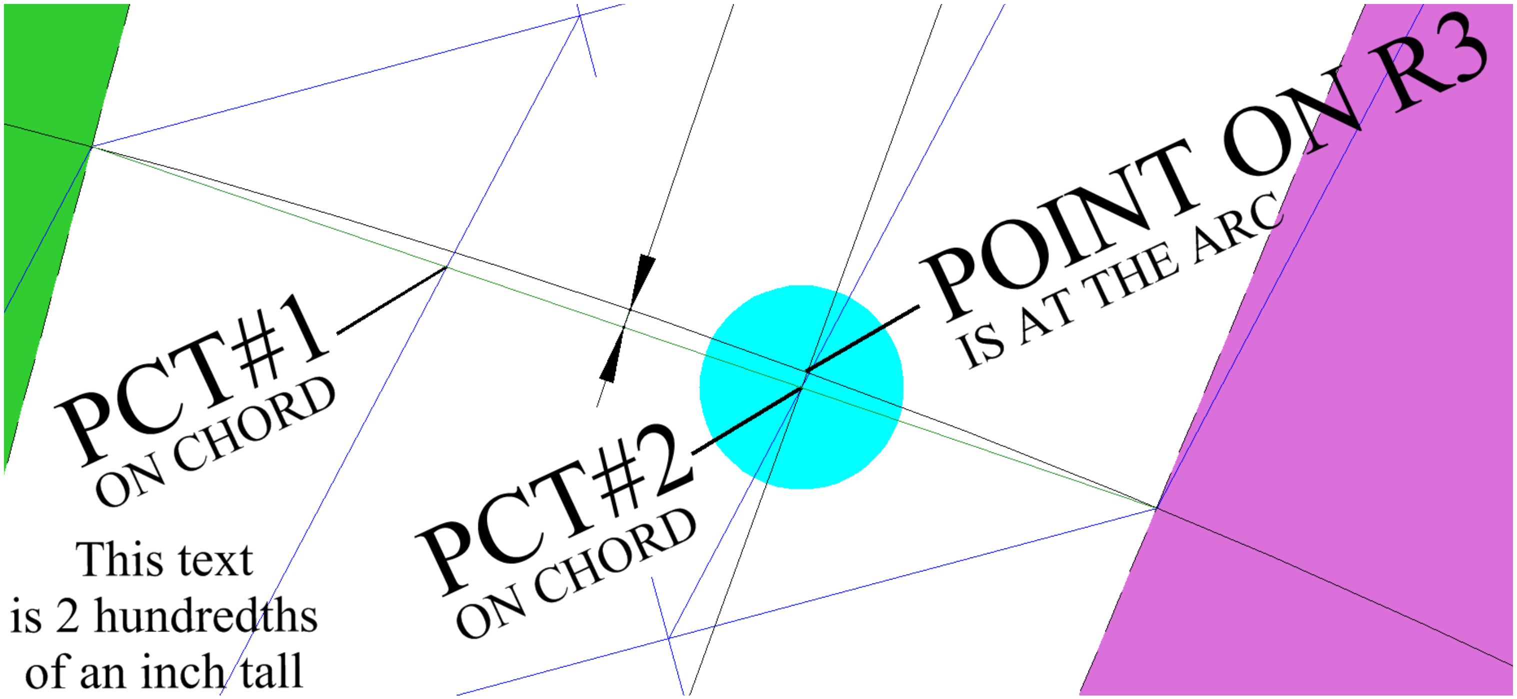
See magification
on following page.

At each step, including along the
trisected arc, Ant-A moved twice
the distance as Ant-B.

Appendix Z - Page 4 - Solution

R1-V-R4 is 120-degrees. R3-V-R4 (a third is) 40-degrees as shown. Mirror R3 about AB14 to show R2.
Need more room to trisect the chord? See Appendix G starting on page 68 for a technique to work further away from V.

Page 5 – Magnification



This image is a magnification of trisection area from the previous page.

CAD hairlines didn't exist in Euclid's world.
The radius of Arc-a was measured at 3.061655 inches.
The chord was measured at 0.398591 inches.

Between the two arrowheads is the sagitta-short measured and calculated at 0.164931 mm, a little thinner than the 0.181 mm of thick human hair. Any line thickness over 0.164931 mm would render the difference between the arc and chord infinitesimal. What line thickness did Euclid have available to use? We don't know.

This is about a 2300% magnification. At 100% the text in the lower left corner is 2/100ths of an inch tall.

Q.E.D. (iterum)

Afterword

In school I was told trisection wasn't possible. I started with sketches, moved to precision drawings on computers, then to CAD. After 52+ years of part time effort will there be any practical impact from this presentation? It may provide motivation for present and future generations to question any statement similar to those in *Yet* on page 4 and perhaps make progress for those that follow.

Was an incandescent lamp possible or not? At the start of his quest Edison[42] didn't know. Two teenagers proved[43] the Pythagorean Theorem using trigonometry, long considered impossible. Me? Trisection of an angle using traditional Euclidean geometry was widely held to have been proven impossible. Why would anyone address a problem unsolved for millennia and "proven" to be unsolvable?

George Bernard Shaw wrote[44]:

> *The reasonable man adapts himself to the world.*
>
> *The unreasonable man persists in trying to adapt the world to himself.*
>
> *Therefore all progress depends upon the unreasonable man.*

Some have opined Shaw is praising "unreasonable" people in general. Not quite. Shaw is noting that those who participate in creating progress share at least one common motivational attribute: a dissatisfaction with the world, or some part of it.

Alfred, Lord Tennyson, wrote *Ulysses*[45] whose last stanza motivated me, especially as I aged:

> *We are not now that strength which in old days*
> *Moved earth and heaven, that which we are, we are;*
> *One equal temper of heroic hearts,*
> *Made weak by time and fate, but strong in will*
> ***To strive, to seek, to find, and not to yield.***

[Bolding is mine.]

One more inspiration:

"We have serious problems to solve,
and we need serious people to solve them.[46]"

There may be multiple ways to address, and hopefully solve, serious problems. Old solutions can be used again or updated then used. There are also new solutions.

The spark of an idea may be the matter of milliseconds. To make that idea perform … often takes time. Inventors may, or not, have an idea what an invention will provide. I'll wager Edison didn't foresee ubiquitous MP3 players inside ubiquitous cell phones when he invented the tin foil phonograph[47].

The benefit of this construction is to remove an improper label on information considered "incorrect". Humans can travel faster than a good horse and really do need more than 64kb on their computer. Something practical? No. I do see one more "not possible" becoming possible.

The list of ideas previously improperly labeled "impossible" grows longer.

Those inspired may create some form of faster-than-light travel.

I hope they call it warp speed.

Endnotes

Start on the next page

[1] Euclid (circa 300 BC) of Alexandria, Egypt was the most prominent mathematician of Greco-Roman antiquity, best known for his treatise on geometry, *the Elements*. Per https://www.britannica.com/biography/Euclid-Greek-mathematician/Renditions-of-the-Elements which also describes the potential for Euclid including the works of others in his publications.

[2] Babylonian land surveys 1900 BC https://theconversation.com/how-ancient-babylonian-land-surveyors-developed-a-unique-form-of-trigonometry-1-000-years-before-the-greeks-163428 also D. F. Mansfield's *Plimpton 322: A Study of Rectangles* Found Sci 26, 977–1005 (2021). https://doi.org/10.1007/s10699-021-09806-0 also at https://link.springer.com/article/10.1007/s10699-021-09806-0

[3] Babylonian use of geometry prior to Euclid per https://www.smithsonianmag.com/science-nature/ancient-babylonians-were-using-geometry-centuries-earlier-thought-180957965/

[4] The three unsolved problems of geometry https://www.encyclopedia.com/science/encyclopedias-almanacs-transcripts-and-maps/three-unsolved-problems-ancient-greece where it says "It became the practice in traditional Greek mathematics to accept geometrical constructions only if they could be performed with an unmarked straightedge and a compass." Notice the word *constructions* which may also be considered a postulate, being *a statement that is accepted as true without proof*. In the *Elements*, Euclid gives five postulates that are the starting points for the propositions or theorems given in the body of the book. The first three of these postulates address the construction of a straight line and a circle:

> A straight line can be drawn between any two points.
> A finite straight line can be extended indefinitely.
> A circle can be drawn with any center point and any line segment as a radius.

Although none of these postulates (or any others) refer directly to a straightedge or a compass, this tradition, usually attributed to Plato (427-347 BC), became an integral part of Greek geometry. The Greeks referred to constructing geometric figures using only a straightedge and a compass as the plane method. Although the bulk of Greek geometry was constructed using plane methods, three problems defied solution by these methods for centuries. [underlining mine, notice "construction" not "proof"]

Per https://www.encyclopedia.com/science/encyclopedias-almanacs-transcripts-and-maps/three-unsolved-problems-ancient-greece

[5] In 1836 Pierre Laurent Wantzel wrote *Recherches sur les moyens de reconnaître si un problème de Géométrie peut se résoudre avec la règle et le compas* It was published the next year in the *Journal de Mathématiques Pures et Appliquées*. 1. 2: 366–372. [italics mine] Available at http://www.numdam.org/item/JMPA_1837_1_2__366_0.pdf as an 8 page PDF in French.

His name is sometimes presented as "M. L. Wantzel". "M." is the abbreviation of "monsieur", a French honorific equivalent to the English "Mr." It appears M. Wantzel preferred his middle name over his first.

An English language version of the French language paper was found at http://bit-player.org/wp-content/extras/trisection/Wantzel1837english.pdf as translated by Brian Hayes October 2006. The title is: *Research on ways to recognize if a geometry problem can be solved with the ruler and compass* A part of the first paragraph in Section IV is reproduced below. The formula is formatted differently from the text provided only by putting fractions within parenthesis.

The word "ruler" appears to indicate a straightedge with divisional markings. I suspect that M. Wantzel was aware that a plain straightedge without such markings was required. It may have been a translation anomaly.

It appears that the procedure as described by M. Wantzel "depends" on a single equation from problem to solution. Consider a solution that uses two different operations in concert to achieve the goal. That possibility makes the statement after the semicolon a logical fallacy.

> The trisection of the angle depends on the equation $x^3 – (3/4)x + (1/4)a = 0$. This equation is irreducible if it has no root that is a rational function of a, and that is the case as long as a remains algebraic; thus the problem cannot be solved in general with ruler and compass.

It may have been he intended to indicate the problem cannot be solved in general with that single equation.

[6] This particular fallacy may be *post hoc ergo propter hoc,* Latin for "after this, therefore because of this." **Post hoc is a logical fallacy in which one event seems to be the cause of a later event because it occurred earlier**. It is not the same as *cum hoc ergo propter hoc* "with this, therefore because of this" in which two events occur simultaneously or the chronological ordering is insignificant or unknown. [italics and bolding mine]

It may also be a ***Bad Reason Fallacy*** **which stems from the claim that because the reason(s) given for a certain conclusion are bad therefore the conclusion must also be incorrect**. This fallacy supposes that it is not possible to give a bad reason for a correct conclusion. It is possible to give bad reasoning for a valid conclusion. Fallacy formula: The reason A given for argument B is bad, therefore conclusion B is not valid. Example of *Bad Reason Fallacy*: Dogs are afraid of heights, therefore dogs don't fly. Though it may be true that dogs are afraid of heights, that is not the reason they do not fly. From https://www.logicalfallacies.org/bad-reason-fallacy.html [bold, italics, and underlining are mine]

It is also possible to give a good reason for an invalid conclusion. Ex: Trisecting an angle could be reduced to solving a particular equation. Since most of those equations could not be solved with unmarked straightedge and compass (TRUE), neither could the trisection problem (FALSE).

Such a statement reaches an erroneous conclusion that no tool, or tools, other than those particular equations could construct an angle trisection.

[7] Wantzel logic error via an allegory

allegory ăl′ĭ-gôr″ē/, a noun with several meanings including: The representation of abstract ideas or principles by characters, figures, or events in narrative, dramatic, or pictorial form. Per *The American Heritage® Dictionary of the English Language, 5th Edition* via https://www.wordnik.com/words/allegory Even simpler: a symbolic representation.

Consider the goal is to put a nail into a board and there are no hammers. Wantzel might have said:

Hammers put nails in boards. I have no hammer; thus nails cannot be put in the board.

Why is the statement after the semicolon false? Because other tools can put nails in boards. Here are a few solutions without hammers.

a) Use your hands to push the nail a little bit into the board. Smack board toward a hard surface such that the nail head impacts first. This puts the nail into the board without a hammer.

b) Put the board and a nail into a vice or grasp them with a c-clamp. Tighten and the screw pushes the nail in to the board. Most vices use a threaded rod as does the c-clamp. The screw is one of the six “classic” or “simple” machines. See https://www.thoughtco.com/six-kinds-of-simple-machines-2699235

Lacking a hammer does not preclude other solutions to putting a nail in the board.

Similarly using specific equations is not the only way to trisect angles.

Wantzel died young and “Ordinarily he worked evenings, not lying down until late; then he read, and took only a few hours of troubled sleep, making alternately wrong use of coffee and opium, and taking his meals at irregular hours until he was married.” Stated in the article “Pierre Laurent Wantzel” by Professor Florian Cajori (1918) in the *Bulletin of the American Mathematical Society*. 24 (7): pages 339–347. doi:10.1090/s0002-9904-1918-03088-7. Found at https://www.ams.org/journals/bull/1918-24-07/S0002-9904-1918-03088-7/S0002-9904-1918-03088-7.pdf (9 page PDF) https://www.ams.org/journals/bull/all_issues.html

[8] National Science Foundation (NSF) *The Surprising Truth Behind the Construction of the Great Pyramids* (May 18, 2007) https://new.nsf.gov/news/surprising-truth-behind-construction-great [in the excerpt below bolding and text in square brackets are mine]

The widely accepted theory — that the pyramids were crafted of carved-out giant limestone blocks that workers carried up ramps — had not only not been embraced by everyone, but, as important, had quite a number of holes. [. . .] the mysteries had actually been solved by Joseph Davidovits, director of the Geopolymer Institute in St. Quentin, France, more than two decades ago [prior to 1987, 20 years before the 2007 article]. Davidovits claimed that the stones of the pyramids were actually made of a very early form of concrete created using a mixture of limestone, clay, lime and water.

A year and a half later, after extensive scanning electron microscope (SEM) observations and other testing, Barsoum and his research group finally began to draw some conclusions about the pyramids. They found that the tiniest structures within the inner and outer casing stones **were indeed consistent with a reconstituted limestone**. The cement binding the limestone aggregate was either silicon dioxide (the building block of quartz) or a calcium and magnesium-rich silicate mineral.

The stones also had a high water content — unusual for the normally dry, natural limestone found on the Giza plateau — and the cementing phases, in both the inner and outer casing stones, were amorphous, in other words, their atoms were not arranged in a regular and periodic array. **Sedimentary rocks such as limestone are seldom, if ever, amorphous**.

The sample chemistries the researchers found do not exist anywhere in nature. "Therefore," says Barsoum, "it's very improbable that the outer and inner casing stones that we examined were chiseled from a natural limestone block." More startlingly, Barsoum and another of his graduate students, Aaron Sakulich, recently discovered the presence of silicon dioxide nanoscale spheres (with diameters only billionths of a meter across) in one of the samples. This discovery further confirms that **these blocks are not natural limestone.** [bolding and text in square brackets are mine]

[9] Trisecting an Angle

Quote and source are below. [bolding, italics, and text in square brackets are mine]

The methods that the Greek mathematicians did find to trisect an angle involved curves such as conic sections or more complicated curves requiring mechanical devices to construct. [. . .] None of these curves, however, could be constructed using the restrictions required by traditional Greek geometry. [true – the ***curves*** could not be constructed. **That does not preclude a solution that does not involve conic sections or more complicated curves.**]

In 1837 Pierre L. Wantzel (1814-1848) completed a proof that the problem was impossible using only a straightedge and a compass. Wantzel essentially showed that trisecting an angle could be reduced to solving a cubic equation. Since most cubic equations could not be solved with straightedge and compass, neither could the trisection problem. **This put a stop to attempts by serious mathematicians to solve the problem.**

From https://www.encyclopedia.com/science/encyclopedias-almanacs-transcripts-and-maps/three-unsolved-problems-ancient-greece The reaction to "impossible" made reviewers scarce. Possible solutions by "amateurs" (because the "serious" had stopped trying) increased the scarcity of reviewers.

What seems to be lacking is the general understanding that a problem which can't be solved one manner does not definitively preclude a solution in another manner.

[10] Iowa State University, Institute for Transportation *Trains: A history* (8/16/2016) https://intrans.iastate.edu/news/trains-a-history/

Germany had a railway in 1550, but it still used horse power. In 1825 the *Stockton & Darlington Railway* opened in Great Britain using steam locomotives on rails. https://www.britannica.com/technology/history-of-technology/Steam-locomotive

The steam engine had been developed earlier to transport coal. In 1825 an improved engine took 450 people 25 miles from Darlington to Stockton at 15 miles per hour. By 1830 a more improved engine could obtain and sustain a speed of 36 miles per hour. https://www.historytoday.com/archive/george-stephensons-first-steam-locomotive

[11] The average horse can run between 25 and 30 miles per hour. Their sprint speed ("at the gallop") reaches about 35 miles per hour for short distances. Per https://www.nationalequine.org/basics/how-fast-horse-run/

[12] Da Vinci did design and demonstrate a hang glider. https://www.leonardodavinci.net/flyingmachine.jsp

[13] In 1878 the Wright Brothers were inspired by their father's present, a small model helicopter of cork, bamboo, paper, with a rubber band to power the blades as designed by French aeronautical pioneer Alphonse Pénaud. 25 years later was the first powered, manned flight by the Wright Brothers was at Kitty Hawk, North Carolina, USA on December 17, 1903 https://www.history.com/topics/inventions/wright-brothers It is widely held they were the first, yet … how about Alberto Santos-Dumont of Brazil and Gustave Whitehead of Connecticut? See https://www.history.com/news/history-faceoff-who-was-first-in-flight

[14] Luft Stalag III, a WWII German prison camp, was built to keep people in. For more see https://www.history.com/news/great-escape-wwii-nazi-stalag-luft-iii A more modern facility was considered escape-proof yet three men escaped from Alcatraz Island penitentiary in 1962. 60^{+} years and they have not yet been found. See https://www.latimes.com/california/newsletter/2022-06-20/escape-alcatraz-prisoners-60-years-later-essential-california France's Maginot Line was built to keep people out. It also failed.

[15] Albert I was a rhesus monkey, shot into space. He isn't well known to history and wasn't the first lost in the attempt. See NASA, *Animals in Space* https://www.nasa.gov/history/a-brief-history-of-animals-in-space/ In

1951 Russia's fifth and sixth launch of two dogs were successful.

[16] On January 31, 1961 astrochimp "Ham" survived a sub-orbital flight on a Redstone rocket. The name "Ham" was an acronym for Holloman Aero Med, a medical department at the Holloman Air Force base https://www.holloman.af.mil/ about six miles southwest of of Alamogordo, New Mexico. The base was re-named for Colonel George V. Holloman, U. S. Army (1902–1946) a pioneer in guided missile research. In 1938 he received the Mackay Trophy as co-inventor of the first aircraft automatic landing system. https://www.nytimes.com/1938/07/22/archives/receive-mackay-trophy-two-army-captains-made-first-automatic.html

FYI: In 1946 military aviation was embodied in the Navy and the Army Air Corps. In mid-1947 the AAC became the separate United States Air Force (USAF) with the implementation of the National Security Act of 1947 see https://web.archive.org/web/20080512165955/http://www.intelligence.gov/0-natsecact_1947.shtml

[17] On April 12, 1962 Vostok 1 took the first human to orbit this tiny blue ball in space. The cosmonaut was Yury (or Yuri) A. Gegarin, See https://www.britannica.com/technology/Vostok-Soviet-spacecraft On May 5, 1961 the Redstone carried Lieutenant Commander Alan B. Shepard, U. S. Navy, the first American in space and first launch of Project Mercury.

Project Mercury https://www.nasa.gov/project-mercury/

Project Gemini https://www.nasa.gov/gemini/

[18] Apollo 8 launched on December 21, 1968 to orbit the moon and return. NASA https://www.nasa.gov/missions/apollo/apollo-8-mission-details/

[19] Apollo 11 was launched July 16, 1969 and achieved lunar orbit. On July 20, 1969 Neil Armstrong (mission commander) was the first to walk on the moon. He was followed by Edwin E. Aldrin Jr. (lunar module pilot). Michael Collins (command module pilot) remained in orbit. NASA, Apollo 11 https://www.nasa.gov/history/apollo-11-mission-overview/

[20] The problem is often presented as two trains starting at different ends of a single track and heading toward each other at different velocities. An added complication is a bee traveling from one train to the other and back again at a third velocity. Where and when do the trains meet? How far does the bee actually fly? See https://mathworld.wolfram.com/TwoTrainsPuzzle.html This trisection solution posits an arc-shaped track known in the railway industry as a "curve".

[21] From Dr. Stephen Richards Covey's *The 7 Habits of Highly Successful People* (1989). The second habit is *Begin With the End in Mind* and can be found at https://www.franklincovey.com/the-7-habits/habit-2/ That website is for the company co-founded by Dr. Covey. The phrase has at least two meanings. The book presents it as a personal planning tool. Know where you want to wind up before starting. Another meaning is in a reverse order: If you know the solution (the "end" to keep in mind) can you reverse-engineer to get to the start of the problem? This led to looking deeply at a solution and what could be derived from that solution to work backwards to the problem.

I had learned this from the crayons and paper place mats to entertain children in restaurants. My parents were vexed when I completed them expeditiously. To save time on such mazes: start at the exit. Starting where it says "Start" leads to a large number of decision points, many of which (eventually) lead to dead ends.

[22] A pivotal moment in the movie *Star Trek: The Wrath of Khan* (ST:TWOK) (1982) is presented in context at this 1m 46s clip https://www.youtube.com/watch?v=j08kI7-T7Vo To see the *Kobayashi Maru Test*

(referenced in the first clip) see the first few minutes of an 8m 58s clip at https://www.youtube.com/watch?v=cU1ah6MOorg

This led to not trying to find the two trisecting rays, one between each third, but looking at half the angle presented and finding only one ray. Using Dr. Covey's *Begin With the End in Mind* (reverse engineering variant described above) the half-a-problem showed two segments, one twice as large as the other. That provided the basis to postulate Ant-A's travels along the arc of the larger segment being twice as long as Ant-B's travels along the arc of the smaller segment. Both are traveling along Arc-a, but starting at separate ends and moving toward each other at different speeds. Ant-A had to travel twice as fast as Ant-B to arrive at the same time at R2. That makes more sense after the Presentation starting on page 6. Seeing three identical angles did not trigger the more visible construct of using one pair of thirds at double the size of one third.

[23] Paleolithic Art is commonly called cave art. It seems reasonable to propose that early humanoids drew in the dirt using their fingers, toes, nose, chin, sticks (hard plant parts), rocks, or bones. Their cave paintings survived and were generally made with either red or black pigment. The reds were made with iron oxides (hematite), whereas manganese dioxide and charcoal were used for the blacks. Per https://www.britannica.com/art/cave-art See also *Prehistoric pigments* from the Royal Society of Chemistry at https://edu.rsc.org/resources/prehistoric-pigments/1540.article

[24] Application of paints may have been by any of the methods described in the endnote above or softer plant parts, early versions of paint brushes. *This includes natural pigments such as ochre and charcoal applied to cave walls by using plants or the artists' hands as brushes*. Per https://www.theartstory.org/movement/cave-art/ [Italics and underlining mine.] Ochre, also written as "ocher", also described as iron ochre, is a mixture of ferric oxide, clay, and sand. This creates pigments in yellows, oranges, browns, and shades of each.

[25] *How to make and use your own reed pen* from https://www.painters-online.co.uk/tips-techniques/mixed-media/articles/how-to-make-and-use-your-own-reed-pen-with-jason-bowyer/ The article was originally features in the April 2018 issue of *The Artist*. Alas, there was no mention of line width. Nor is there a reference to line width in *Making Reed Pens* by James W. Grimes, Associate Professor of Painting, the Ohio State University (2013) 44. 18-19. 10.1080/00119253.1943.10742141. It was included in https://www.courses.shtyrmer.com/srisa/wp-content/uploads/2018/06/Lesson-4-Reed-Pen-making.pdf

[26] The line width of reed pens is not precisely known as surviving writing shrunk over the centuries. The first use of reed pens was in the 4th century BC by scribes in Ancient Egypt writing on papyrus per http://www.historyofpencils.com/writing-instruments-history/history-of-reed-pen/ It seems reasonable to speculate that early reed pens created a wider line than the nibs that followed.

Based on findings at Saqqara in Egypt, Steven Roger Fischer suggests that the reed pen might well have been used for writing on parchment as long ago as the First Dynasty, or around 3,000 BC. From his *A history of writing* (2001) at https://archive.org/details/historyofwriting0000fisc

see also *A history of writing : from hieroglyph to multimedia* (2002)
Edited by Anne-Marie Christin https://archive.org/details/historyofwriting0000unse_j0t3

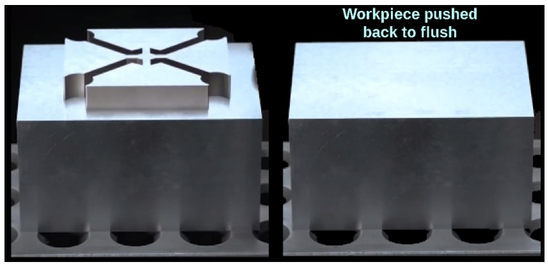
[27] Quill pen data included 11 parameters that did not include line weight.
https://www.vintagepens.com/pen_measurements.shtml
and https://www.vintagepens.com/pendata.pdf

[28] Nibs — From a May 19, 2010 post: "My Lamy Safari's 1.1 mm italic nib produces a line width of about .7 to .8 mm with Noodler's Baystate Blue ink. The nib itself appears to be 1.0 mm wide. With a little practice and a magnifier, you should be able to divide 1 mm scale divisions into .1 mm increments (inkrements!) with

reasonable consistency." Post from https://www.fountainpennetwork.com/forum/topic/155503-how-do-you-measure-line-width/ retrieved 4/16/2024 [underlining mine. The poster wrote "inkrements"]

0.1mm line width from a nib also reported at https://www.reddit.com/r/fountainpens/comments/14ygdtt/can_anyone_estimate_the_line_width_if_this_nib_i/ retrieved 4/16/2024

[29] Wood shop – given a 4" x 4" wood square saw it in half to get two pieces 2"x4". Nope. The material removed by the saw is *kerf*, the blade's cutting width. Depending on the "set" of the teeth at a small angle away from the saw blade's body, the kerf will be wider than the body of the saw blade. Water jet cutters take less kerf. Laser cutters can take even less.



Wire Electrical Discharge Machining (Wire-EDM) can take kerf of 0.0008 inches (0.02032mm) and smaller for an ***infinitesimal*** kerf. The kerf is known to exist, but is smaller than our ability to observe. These two images are from https://www.facebook.com/reel/327530747027116

[30] I had to be careful in Engineering Graphics when using a freshly sharpened 8H pencil. Some exercise problems were given to us on soft pulpy paper. With too much pressure the 8H pencil cut the paper.

[31] **discrete** /dĭ-skrēt′/ is an adjective with multiple meanings including

Defined for a finite or countable set of values; not continuous.

From *The American Heritage® Dictionary of the English Language*, 5th Edition on line at https://duckduckgo.com/?t=ftsa&q=define+discrete

[32] Re definition and examples of recurring/repeating numbers see *Recurring Decimal Definition* (Illustrated Mathematics Dictionary) by Rod Pierce (27 Aug 2023) from *Math Is Fun*. Retrieved 16 Apr 2024 from http://www.mathsisfun.com/definitions/recurring-decimal.html

[33] **infinitesimal** /ĭn″fĭn-ĭ-tĕs′ə-məl/, is an adjective meaning

Immeasurably or incalculably minute.
Capable of having values approaching zero as a limit. [<< This one applies here]
Infinitely or indefinitely small; less than any assignable quantity or value; very small.

From *The American Heritage® Dictionary of the English Language*, 5th Edition at https://duckduckgo.com/?t=ftsa&q=define+infinitesimal

[34] Re two train problem: see endnote on page 92. [< Click page number to go direct to appropriate end note]

[35] At the bottom of the presentation's title page are two characters referenced in the presentation. At the lower left is "Ant-A", the animated character *Atom Ant*, created by Hanna-Barbera in 1965, and shown 1965 to

1968. At the lower right is “Ant-B”, based on *Aunt Bee*, a character played by Frances Bavier in *The Andy Griffith Show* (1960–1968) and the spinoff *Mayberry R.F.D*. (1968–1971).

[36] The text cited is the opening text of the *Bee Movie* (2007) as written by Jerry Seinfeld et. al.

see https://www.imdb.com/title/tt0389790/ The text refers to the time when analysis of flight forces were based on fixed wing and rotary wing aircraft, possibly dating back to the 1930s when (story goes) an aerodynamicist was talking to a biologist and took the opportunity to ask about the flight of bees. The aerodynamicist concluded that bees should not be able to fly at all.

That potentially apocryphal analysis didn’t take into account two factors. 1) Bee wings are flexible and can change camber in flight. While rotary wings can change pitch angle during rotation they can’t change camber. 2) When in flight bee wings flap up and down. Unlike birds or fixed wing aircraft, bees do not glide very well with rigid wings regardless of camber.

[37] It is (still) widely held that Galileo said “It still moves” when exiting the court. That probably isn’t true. https://www.scientificamerican.com/blog/observations/did-galileo-truly-say-and-yet-it-moves-a-modern-detective-story/ The earliest mention of the phrase in print was in a book entitled *The Italian Library* (1757) by Italian author Giuseppe Baretti. See URL for the full story.

[38] Burning lenses were known to the Greeks about 420 BC, but they were not for vision enhancement. Ancient Egyptians [circa 3100 BC to circa 1 BC] have produced glass lenses that were once readily identified as reading aid lenses, but this theory is now largely discredited because the surviving objects *are* lenses — but not considered suitable as reading aids since they are too thick and of inferior material to be usable.

Per https://history.stackexchange.com/questions/41312/what-evidence-is-there-of-the-vision-aids-people-used-before-the-invention-of-ey citing *Das alte Ägypten* W. Foy (Ed.): Kulturgeschichtliche Bibliothek (= I. Reihe: Ethnologische Bibliothek). Band 2. C. Winters, Heidelberg 1920, S. 412.) Text in [square brackets] above are mine. Italics were in the text. German language version available at https://archive.org/details/dasaltegypten00wied

[39] The *Faber-Castell Ecco Pigment* – according to the manufacturer the Ecco is a mechanical pencil using a 0.1 mm diameter lead. It was not available to Euclid. It is on Amazon at https://www.amazon.com/Faber-Castell-Pigment-Mechanical-Pencil-Pencils/dp/B005EFHVBM As of 5/11/2024 it had six reviews with more one-star ratings than any other rating. (zero star ratings are not allowed) There are comments including this is not a mechanical pencil but a thin tipped marker. Euclid probably didn’t have those either.

[40] From *Ram’s theorem for Trisection* by Ramachandra Bhat on 10 February 2019 where he paraphrases Wantzel in http://arxiv.org/pdf/1902.03592 On page 2 Bhat wrote:

While many Geometers have been exploring to find solutions to these problems, in 1837, Pierre Wantzel proved that finding a solution to the problem of trisection of an angle is impossible [Bhat end notes: 1, 2]. This only meant, using only ruler and compasses, it is not possible to trisect an angle of any given value, *with the tools and knowledge available at that instant of time*. It should also be noted that the proof of impossibility considers primarily the constructability of the angle of value equivalent to one-third of the given value and not the trisectability of the given angle directly. [in this paragraph italics appear in Bhat’s text, see link above]

2. The problem definition: […] This means […] Since the measure of the given angle was unknown, Greeks could only explore the trisectability and not constructibility. Hence, the definition by the Euclidean geometry may be presented as:

Definition 2.1. "**Divide the given angle** *(of unknown value)* i**nto three equal angles** *(angular parts)* ~~or construct an angle equal to one-third of the given angle (of unknown value?)~~ *using only two tools, viz., (i) an unmarked straight edge (ruler) and (ii) a compasses"* [in 2.1 above, the parentheses, bold, italic and strikethru appear in Bhat's text. Bhat does not refer to a writing tool and something to write upon]

[41] Measurements for the diameter of human hair mentioned in the presentation and appendices were taken from *The Physics Factbook*, an encyclopedia of scientific essays. Diameters range from 17–50 µm (flaxen hair) to 56–181 µm (coarse black hair) per https://hypertextbook.com/facts/1999/BrianLey.shtml 1 millimeter (mm) = 1000 micrometers (µm) so the diameter of human hair ranges from 0.017 mm (thin hair) to 0.181 mm (thick hair).

[42] Edison made many attempts at creating an incandescent lamp (bulb). None had worked and point of view is important in understanding the benefit of continuing the effort. W. S. Mallory, a close associate, said: "Isn't it a shame that with the tremendous amount of work you've done you haven't been able to get any results?" Edison replied: "**Results! Why, man, I have gotten a lot of results! I know several thousand things that won't work.**" [bolding mine]

The quote appears in *Edison: His Life and Inventions* (1910) by Frank Lewis Dyer and Thomas Commerford Martin in Volume 2, page 616 in Chapter 24: *Edison's Method in Inventing.* This link is direct to the quote on page 616 of 989 https://books.google.com/books?id=B7A4AAAAMAAJ&q=%22gotten+a+lot Specific numbers such as 10,000 ways etc. creep in with frequent retelling of stories. See https://quoteinvestigator.com/2012/07/31/edison-lot-results/

Why did I keep trying after 52 years of part time effort? Edison also said: "**Many of life's failures are people who did not realize how close they were to success when they gave up**." From https://www.thomasedison.org/edison-quotes [bolding mine]

[43] April 10, 2024 At an American Mathematical Society meeting, two high school students presented a proof of the Pythagorean theorem that used trigonometry, an approach previously considered impossible https://www.scientificamerican.com/article/2-high-school-students-prove-pythagorean-theorem-heres-what-that-means/

[44] From Shaw's *Maxims for Revolutionists* (1903) about 20 pages, a collection of short thoughts organized under multiple topics. Some topics have one line, others more. Available at https://www.gutenberg.org/cache/epub/26107/pg26107.html

[45] The poem *Ulysses* by Alfred, Lord Tennyson is available on line at https://www.poetryfoundation.org/poems/45392/ulysses

[46] From *The American President* (1995) written by Aaron Sorkin and directed by Rob Reiner. The quote can be read at https://www.imdb.com/title/tt0112346/quotes The quote is part of a 4m 49s speech on line at https://www.youtube.com/watch?v=OC2jhQ0KAAU.

[47] Early Sound Recording Devices - During the early 1880s a contest developed between Thomas A. Edison and the Volta Laboratory team of Chichester A. Bell and Charles Sumner Tainter. The objective was to transform Edison's 1877 tinfoil phonograph, or talking machine, into an instrument capable of taking its place alongside the typewriter as a business correspondence device. This involved not only building a better machine, but finding a substance to replace the foil as the recording medium. From https://www.loc.gov/collections/emile-berliner/articles-and-essays/gramophone/

Un-numbered Endnotes

[a] **6.25%** In the Ray-A and Ray-B example introduced on page 6 of the Appendix B − *Infinitesimal Geometry* the angle R1-V-AB14 is 75°. The angle RA-V-RB, is 4.6875° or 6.25% of the R1-V-AB14 angle. That same 6.25% appears again as the zone of solution on page 5 of Appendix C − *Range Narrowing* being the difference between R62.5% and R68.75%. On page 2 of Appendix Z − Simpler the 6.25% zone of solution appears again. In the Presentation Part 2 the R0%-V-R100% is 30°. R62.5%-V-R68.75% (zone of solution) is 1.875000° or 6.25% of 30°.

1**.5625%** On page 3 of Appendix F − *Angles >180° & 3rd Tool* the zone of solution" is not 6.25% of the half problem (the original problem being R1-V-R4, the half problem R1-V-AB14) but 1.5625% being R67.1875% less R65.625% because of the "third set of tools".

Both Angles When working with the whole angle R1-V-R4 as shown on page 2 of Appendix Y − *Why oh Why* the same 6.25% zone of solution appears again. After shifting to the whole 270° angle on page 3 of Appendix Y − *Why oh Why* the R65.625%-V-R67.1875% (zone of solution) is 4.218750° = 1.5625% of 270°. 1.5625% is one half of 3.125% which is one half of 6.25%. 1.5625% is one quarter of 6.25%, the result of two additional bisections.

Why do these zones of solution re-occur? Because the postulate is based on a construction of rays as percentages of the angle under consideration. The zones of solution appear as percentages of an unknown angular value. When using two ant-steps the zone of solution is 6.25% of the angle under consideration. Using the third pair of tools the zone of solution is 1.5625% of the angle under consideration.

[b] So why use half the problem angle instead of the whole angle? While the zones of solution are the same percentage after using the ant's two-step (6.25%) or three-step (1.5625%), the chords are different. As a matter of historical note the definition of "three equal thirds" hindered alternative points of view.

[c]

What did Ant-A say when it took the first step from R0% to R50%?

One giant step for an ant.

One great leap for humankind!